Math 489 Midterm I  
Oct. 10, 2019

1. (45 points) Consider the dynamical system $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2 + ax$.
   (a) (5 points) Find all fixed points.
   (b) (10 points) Determine the stability of the fixed points for all values of $a \in \mathbb{R}$ except \pm 1 and 3.
   (c) (5 points) Describe all orbits when $a = 1$. What kind of fixed point is $x = 0$ in this case?
   (d) (10 points) Find the range of parameters $a$ for which the dynamical system has a 2-periodic orbit. Then find the range of parameters $a$ for which this orbit is a sink.
   (e) (5 points) Describe as precisely as you can what happens to the orbits that start near zero when the parameter $a$ is slightly smaller than $-1$.
   (f) (10 points but difficult) Describe all orbits when $a = -1$. What kind of fixed point is $x = 0$ in this case?

2. (20 points) Let $A = \begin{pmatrix} -3.75 & -4.5 \\ 2.25 & 3 \end{pmatrix}$ show that its eigenvalues are $0.75$ and $-1.5$ and the corresponding eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.
   (i) (5 points) Find $\lim_{n \rightarrow \infty} A^n \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.
   (ii) (5 points) Find $\lim_{n \rightarrow \infty} A^n \begin{pmatrix} -3 \\ 3 \end{pmatrix}$.
   (iii) (5 points) Find $\lim_{n \rightarrow \infty} A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
   (iv) (5 points) What kind of fixed point is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for the dynamical system $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $g(x) = Ax$?

3. (15 points) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by:
   \[ f(x, y) = (-3.75x - 4.5y - (x + 2y)^2, 2.25x + 3y + (x + 2y)^2) \]
   (a) (5 points) Show that $(0, 0)$ is a fixed point and determine its type.
   (b) (5 points) Show that $f(x, -x) = [0.75 - x](x, -x)$, for all $x \in \mathbb{R}$ and that the segment $y = -x$, $-0.25 < x < 1.75$, is part of the stable manifold for the $(0, 0)$ fixed point.
   (c) (5 points) Show that $f(2y, -y) = -1.5(2y, -y)$, for all $y \in \mathbb{R}$ and determine the unstable manifold for the $(0, 0)$ fixed point.

4. (30 points) Consider the dynamical system $g : [0, \frac{1}{10}] \rightarrow [0, 1)$, $g(x) = 10x \text{ mod } 1$. Note that $g$ is smooth except at $0.1, 0.2, \ldots \frac{9}{10}$. Statement continues on the next page.
(a) (2 points) Show that the orbit of \( x \in [0, 1) \) contains a point of discontinuity for \( g \) if and only if there is \( k \in \mathbb{N} \) such that \( x = .x_1x_2 \ldots x_k \) i.e., in decimal representation, all digits after the \( k^{th} \) are zero.

(b) (8 points) Show that if the orbit of \( x \) is periodic then \( x = .x_1x_2 \ldots x_kx_1x_2 \ldots x_k \ldots \) i.e. the digits in the decimal representation of \( x \) repeat. Show that all periodic orbits are repelling.

(c) (10 points) Show that any asymptotically periodic orbit must be eventually periodic. Then show that the orbit of \( x \) is eventually periodic if and only if \( x \) is rational.

(d) (10 points) Show that any point in \([0, 1]\) has sensitive dependence on data under this dynamical system. (If you pick a particular point and show it has sensitive dependence on data you will get half the credit!)
1. (a) \( f(r) = r \Rightarrow r^2 + ar = r \Rightarrow r = 0 \) or \( r + a = 1 \)
   \[ \Rightarrow \begin{cases} p_1 = 0 \text{ and } p_2 = 1 - a \end{cases} \]

(b) \( f'(x) = 2x + a \).
   \[ f'(p_1) = f'(0) = a \]

For \( a \in (\frac{1}{2}, 1) \cup (1, \infty) \) \( p_1 = 0 \) is an unstable and attracting fixed point.

For \( a \in (-\infty, 1) \), \( p_1 = 0 \) is a stable and attracting fixed point.

\[ f'(p_2) = f'(1 - a) = 2 - 2a + a = 2 - a \]

For \( a \in (-\infty, 1) \cup (3, \infty) \) \( p_2 = 1 - a \) is an unstable and repelling fixed point.

For \( a \in (1, 3) \) \( p_2 = 1 - a \) is a stable and attracting fixed point.
1 (c) $f(x) = x^2 + x$.  \[ P_1 = P_2 = 0, \quad f'(0) = 1 \]

\[ f(x) > x \forall x \neq 0. \]

If $x_0 > 0$ we have $x_0 < f(x_0) < f(f(x_0)) < \ldots$

the orbit is increasing and diverging to $+\infty$ (since there is no fixed point positive fixed point).

If $-1 \leq x_0 < 0$ we have $x_0 < f(x_0) \leq f(f(x_0)) \leq \ldots < 0$

the orbit is increasing and converging to 0, since 0 is the only fixed point.

If $x_0 \leq -1$ we have $x_0 < -1 < 0 < f(x_0) < f(f(x_0))$

and the orbit is increasing and diverging to $+\infty$

as in the first case.

\[ (d) \quad f(f(x)) = \frac{1}{2} \left( f(x) \right)^2 + a f(x) = \left( x^2 + ax \right)^2 + a(x^2 + a) \]

\[ = x^4 + 2ax^3 + (a^2 + a)x^2 + a^2 x \]

\[ f''(p) = p \Rightarrow p^4 + 2ap^3 + (a^2 + a)p^2 + a^2 p = p \]

\[ \Rightarrow p = 0 \text{ or } p^3 + ap^2 + (a^2 + a)p + a^2 - 1 = 0 \]
(d) cont.

\[ p > 0 \text{ or } (p + a - 1)(p^2 + (a + 1)p + a + 1) = 0 \]
\[ p_1 = 0, p_2 = 1 - a, \text{ or } p^2 + (a + 1)p + a + 1 = 0 \]

\[ \Delta = (a + 1)^2 - 4(a + 1)(a + 1)(a - 3) > 0 \]
\[ \Rightarrow a \in (-\infty, -1) \cup (3, \infty) \]

So a 2-periodic orbit exists \( \Rightarrow a \in (-\infty, -1) \cup (3, \infty) \)
and is given by
\[ p_3 = \frac{-(a + 1) + \sqrt{(a + 1)(a - 3)}}{2} \]
\[ p_4 = \frac{-(a + 1) - \sqrt{(a + 1)(a - 3)}}{2} \]

\[(f^2)'(p_3) = f'(p_4)f'(p_3) = (2p_4 + a)(2p_3 + a) \]
\[ = 4p_3p_4 + 2a(p_3 + p_4) + a^2 \]
\[ = 4(a + 1) + 2a(-a - 1) + a^2 \]
\[ = -a^2 + 2a + 4 = -(a - 1)^2 + 5 \]

\[(f^2)'(p_3) \in (-1, 1) \quad \Rightarrow \quad -6 < -(a - 1)^2 < -4 \]
\[ \leq \quad 4 < (a - 1)^2 < 6 \]
\[ \Rightarrow \quad 2 < |a - 1| < \sqrt{6} \]

\[ \Rightarrow \quad \{ \begin{array}{l}
3 < a < 1 + \sqrt{6} \\
0 < a < 1 - \sqrt{6} \end{array} \]
\[ \therefore \quad 1 - \sqrt{6} < a < -1 \]
1. (c) The fixed point $p_1 = 0$ has $f'(p_1) = a < -1$ and is a source, while the 2-periodic orbit

$$p_{3,4} = -\frac{(a+1) \pm \sqrt{(a+1)(a-3)}}{2}$$

has $(f^2)'(p_3) = (f^2)'(p_4) = -(a-1)^2 + 5 < 1$ and is a sink. Since $a < -1$, we expect these orbits to approach the fixed points $p_3 < 0 < p_4$ as $x_0$ moves away from zero and approach the 2-periodic orbit. This can be shown using $f^2$.

\[ y = 1 \quad \text{so if} \quad 0 < x_0 < p_4 \quad \text{we have} \quad 0 < x_0 < f^2(x_0) < f^4(x_0) < \cdots < p_4 \]

and this sequence converges to $p_4$ as there are no other fixed points of $f^2$ in $(0, p_4)$.

And for $0 < x_0 < p_4$, we have $p_3 < f(x_0) < 0$ and

$$f(x_0) > f^2(f(x_0)) > f^4(f(x_0)) > \cdots > p_3$$

with the sequence converging to $p_3$, the unique fixed point of $f^2$ in $[p_3, 0)$. Similarly, $p_3 < x_0 < 0$ implies $x_0 > f^2(x_0) > f^4(x_0) > p_3$ and $0 < f(x_0) < f^3(x_0)$. \[ \]
1. (f) The graph of \( f(x) = x^3(x-2) + x \) is

Hence \( f(x) \) is not decreasing at \( 0 < x_0 < 2 \) because

at \( 0 < x_0 < 2 \) decreases to 0;

at \( -1 < x_0 < 0 \) increases to 0 or become positive and decreases to 0;

at \( 2 < x_0 \) increases to +\( \infty \);

at \( x_0 < -1 \) increases to +\( \infty \).

Since the graph of \( f(x) = x^2 - x \) to

then \( x_0 \in (0, 2) \Rightarrow f(x_0) \in (0, 2) \)

and the \( f^2 \) orbit of \( f(x_0) \) decreases to zero

\( x_0 \in (-1, 0) \Rightarrow f(x_0) \in (0, 2) \)

All in all, the orbit of \( x_0 \) under \( f \) converges to zero for \( -1 < x_0 < 2 \); converges to 2 if \( x_0 \in \{-1, 2\} \) and diverges to \( +\infty \) otherwise.

2. \( \begin{bmatrix} A - 0.75 I \end{bmatrix} [1] = \begin{bmatrix} -4.5 & -4.5 \\ 2.25 & 2.25 \end{bmatrix} [1] = 0 \)

\( \Rightarrow 0.75 \) is an eigenvalue with eigenvector \( [1] \)
2. (cont) Similarly,
\[
\begin{bmatrix} A + 1.5 \mathbf{I} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2.25 & -4.5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 0
\]
and \(-1.5\) is an eigenvalue with eigenvector \(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\).

\(\text{(a)}\) \quad A^n \begin{bmatrix} 4 \\ -2 \end{bmatrix} = A^n \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} (-1.5)^n \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix}

The limit does not exist, the sequence diverges to \(\pm \infty\) along the line \(x = -2y\).

\(\text{(ii)}\) Since \(\begin{bmatrix} -3 \\ 3 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -1 \end{bmatrix}\), we have \(A^n \begin{bmatrix} -3 \\ 3 \end{bmatrix} = (0.75)^n \begin{bmatrix} -3 \\ 3 \end{bmatrix}\) converges to 0.

\(\text{(iii)}\) \quad \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \end{bmatrix} = A^n \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} (-1.5)^n \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix}
\quad + (0.75)^n \begin{bmatrix} -3 \\ 3 \end{bmatrix}\) So \(A^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}\) approaches the line \(x = -2y\) but diverges to \(\pm \infty\).

\(\text{(iv)}\) It is a stable since one eigenvalue has magnitude between 0 and 1, the other is over 1.
3. (a) \( f(0,0) = \left( -0, 0+0^2, 0+0+0^2 \right) = (0,0) \)

\[
\begin{bmatrix}
-3.75 + 2(x+2y) & -4.5 - 4(x+2y) \\
2.25 + 2(x+2y) & 3 + 4(x+2y)
\end{bmatrix}
\]

\( \det f(x,y) = \begin{bmatrix} -3.75 & -4.5 \\ 2.25 & 3 \end{bmatrix} \)

hence \((0,0)\) is a saddle fixed point, see Pr. 2.

(b) \( f(x,y) = \begin{bmatrix} -3.75 & -4.5 \\ 2.25 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -(x+2y)^2 \\ (x+2y)^2 \end{bmatrix} \)

for \((x,y) = x \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) we get \( f(x,y) = 0.75 \begin{bmatrix} x \\ -x \end{bmatrix} + \begin{bmatrix} -x^2 \\ x^2 \end{bmatrix} \)

\( = \begin{bmatrix} 0.75-x \\ -x \end{bmatrix} \). So the line \( y=-x \) is invariant under \( f \) and points along the segment \( y = -x, -0.25 < x < 1.75 \) move.

\( |f(x_0,-x_0)| < \left| \frac{0.75-x_0}{1} \right| \leq 1 \)

and their orbit converges to \((0,0)\), the segment is part of the stable manifold.
3. (6) \( f^{-1}(2y, -y) = y \begin{bmatrix} -3.75 & -4.5 \\ 2.25 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -0.5 \\ 0.2 \end{bmatrix} \)

\[ = (-1.5) \begin{bmatrix} 2y \\ -y \end{bmatrix} = (-1.5)(2y, -y). \]

So \( f^{-1}(2y, -y) = -\frac{2}{3}(2y, -y) \)

\( \Rightarrow \) The line \( x = -2y \) is invariant under \( f^{-1} \)

and the orbits starting near it converge under \( f^{-1} \) to \( O_2 \)

(al a rate \( 2/3 \)). So the line is part of the unstable manif.

But any other \( (x_0, y_0) : \quad f^{-1}(x_0, y_0) \rightarrow O_2 \)

must land on the local unstable manifold at \( (0, 0) \), which is a cone tangent to \( \begin{bmatrix} 2 \\ -1 \end{bmatrix} \) to the line \( x = -2y \). By uniqueness, this curve is a small sequence \( x = -2y, y \leq \varepsilon \) \( \Rightarrow \)

\( (x_0, y_0) \in f^{-1} \) \( \Rightarrow \) \( (\text{segment}) \subseteq \{ (x, y) \mid x = -2y \} \).

So \( (x_0, y_0) \in \) line \( x = -2y \Rightarrow \) the line is the unstable manifold.

4. (a) Note that \( g^1(x_1, x_2, ...) = x_2x_3 \ldots \) hence if \( g^{k-1}(x_1, x_2, ..., x_{k-1}) \in \{0.1, 0.2, ..., 0.9\} \) then

\( g^k(x_1, x_2, ..., x_{k-1}) \) \( = 0 \Leftrightarrow x_{k+1}x_{k+2} \ldots = 0 \) \( \square \)
4 (b) If \( X = \ldots x_k x_{k+1} x_{k+2} \ldots \) and \( g^k(x) = x \), then
\[ x_k x_{k+1} x_{k+2} \ldots = x_k x_{k+1} x_{k+2} \ldots \]

\[ x_k x_{k+1} x_{k+2} \ldots x_{2k} = x_k x_2 x_k \]

\[ x_{2k+1} x_{2k+2} \ldots x_{3k} = x_{2k+1} x_{2k+2} \ldots x_{3k} = x_1 x_2 x_k \]

\[ x_{3k+1} x_{3k+2} \ldots x_{4k} = x_{3k+1} x_{3k+2} \ldots x_{4k} = x_1 x_3 x_k \]

\[ x_{4k+1} x_{4k+2} \ldots \]

\[ x = x_1 x_2 \ldots x_k x_1 x_2 \ldots \]

If \( x = 0 \) then \( g^1(0) = 10 > 1 \) and is repelling.
If \( x = \ldots x_k x_{k+1} \ldots \) then its orbit never reaches the discontinuity points of \( g \), see (a), and
\[ \left( g^k \right)^{1}(x) = g^1(g^{k-1}(x)) \cdot g^1(g^{k-2}(x)) \cdots g^1(x) \]
\[ = 10 \cdot 10 \ldots 10 = 10^k > 1 \] and is repelling.

(c) If the orbit of \( x \) approaches \( \{ p, g(p), g^2(p), \ldots \} \) with \( g^k(p) = p \) then \( \left( g^k \right)^{1}(x) \) approaches one of the points on the periodic orbit, say \( p \). But \( p \) is a repelling fixed point for \( g^k \), see part (b). So
\[ \exists \varepsilon > 0 : \forall y \in (p - \varepsilon, p + \varepsilon), y \not= p \text{ implies } \exists M > 0 : \left| g^k(x) - p \right| > \varepsilon \]
Since, by assumption, \((g^k)^u(x) \rightarrow p \in \mathbb{N}\):

\[(g^k)^u(x) - p| < \epsilon \quad \text{and} \; n \geq n_0.

Let \(y = (g^k)^{n_0}(x)\). Using the conclusion then

\[y = p\] otherwise we have the contradiction

\[(g^k)^{n_0}(y) = (g^k)^{n_0}(x) \text{ is both inside and outside the interval} \ (p-\epsilon, p+\epsilon).

So the orbit is eventually periodic.

Let \(x = x_1 x_2 \ldots x_k x_{k+1} \ldots \) be an eventually periodic orbit. Then for all \(x \in \mathbb{N}\):

\[g^i(x) = p, \quad p \text{ is periodic}, \quad \text{i.e.,} \quad \exists \, x_1 x_2 \ldots x_l(x_1 x_2 \ldots x_l)^r \in \mathbb{N}, \quad \text{for all} \; i \geq l.

\[x = x_1 x_2 \ldots x_l \underbrace{(x_{l+1} x_{l+2} \ldots x_{2l})}_{\text{rep}} \ldots \]

\[10^{(k-1)}x = 10^k x - u \quad 10^{k-1}(10^k x - u) - u = p.

\[x = \frac{u}{10^{k-1}} \quad \text{and} \quad 10^k x - u = \frac{u}{10^{k-1}} \quad \text{or} \quad x = \frac{u}{10^k} + \frac{u}{10^k(10^k - 1)} \in \mathbb{Q}.

\[x = \frac{u}{10^{k-1}} \quad \text{and} \quad 10^k x - u = \frac{u}{10^{k-1}} \quad \text{or} \quad x = \frac{u}{10^k} + \frac{u}{10^k(10^k - 1)} \in \mathbb{Q}.

[\text{else}] \quad (10^{k-1} - 1)u = u \quad \Rightarrow \quad p = \frac{u}{10^{k-1}} \in \mathbb{Q} \quad \text{since} \; u \in \mathbb{N}.
4(c). Any point on a periodic orbit is sensitive to dependence on data since it is a solution for an iterate of $g$.

More generally, \( \forall \in \mathbb{R}, \forall \mathcal{O}, \exists \delta > 0 \),

\( \exists x_0 \in \mathcal{O} \) : \( |x_0 - x| < \delta \) and \( |g^k(x_0) - g^k(x)| > \frac{1}{2} \)

Indeed \( \exists x \in \mathcal{O} \) such that

\( g^k(x) = \cdots x_0 x_1 \cdots x_k \cdots \)

\( \exists \delta > 0 \) choose \( k \in \mathbb{N} : 10^{-k} < \delta \). Then any

\( x = x_0 x_1 \cdots x_k x_{k+1} \cdots \)

\( x = x_0 x_1 \cdots x_k x_{k+1} \cdots \)

such that the property \( |x - x_0| \leq \frac{a}{10^{k+1}} + \frac{a}{10^{k+2}} + \cdots \in \frac{a}{10^{k+1}} \left( 1 - \frac{1}{10} \right) = 10^{-k} \).

and we have \( g^k(x_0) = \cdots x_0 x_1 \cdots \in \mathcal{O} \)

\( g^k(x) = \cdots x_{k+1} x_{k+2} \cdots \)

\( g^k(x_0) \in \mathcal{O} \).

So if \( g^k(x_0) \in \mathcal{O}, \frac{1}{2} \) we choose \( x_0 x_1 x_2 \cdots \in \mathcal{O} \).

\( g^k(x_0) \in \mathcal{O}, g^k(x_0) \frac{1}{2} \)

if \( g^k(x_0) \in \left( \frac{1}{2}, 1 \right) \) we choose \( x_{k+1} x_{k+2} \cdots \in \mathcal{O} \).