§ 1.7–8. Sensitive dependence on data, Liouville's

Def. \( f: S \rightarrow S, \ S \subseteq \mathbb{R}^n \). A point \( x_0 \in S \) has sensitive dependence on initial condition if \( \exists \ d > 0 : \ \forall \ N \Subset (x_0) \) neighborhood of \( x_0 \), \( \exists x \in N \) such that \( \| f^k(x) - f^k(x_0) \| > d \).

Example. \( f: [0,1] \rightarrow [0,1] \)

\( f(x) = 3x \mod 1 = \text{fractional part of } 3x \).

Proposition. \( \forall x_0 \in [0,1], \ \forall \delta > 0 \)

\( \exists x \in [0,1], \ |x-x_0| < \delta \) and \( k \in \mathbb{N} : \)

\( \| f^k(x) - f^k(x_0) \| > \frac{1}{2} \)

Remark. The proposition shows all points \( x_0 \in [0,1] \) have sensitive dependence on data for this dynamical system. Later (chapter 3) we will see that all orbits are actually chaotic!

Proof. Any \( x \in [0,1] \) can be written

\( x = \frac{x_1}{2} + \frac{x_2}{2^2} + \cdots + \frac{x_k}{2^k} + \cdots \).
where \( x_j \in \{0, 1, 2\} \ \forall J \), i.e.

\[
x = x_1 x_2 \ldots x_n \quad \text{in base 3}
\]

Note that the orbit

\[
x, f(x), f^2(x), \ldots \quad \text{for } x = x_n x_{n-1} \ldots x_1
\]

is \( x, x_2 x_1, x_3 x_2, \ldots x_3 x_2 \).

Fix \( x_0 = x_n^0 x_{n-1}^0 \ldots x_1^0 \) and \( \epsilon > 0 \)

choose \( k \) such that

\[
\frac{1}{3^k} < \epsilon
\]

and

\[
x = x_n^0 x_{n-1}^0 \ldots x_1^0 x_{k+1}^0 x_{k+2}^0 \ldots
\]

such that

\[
x_{k+i} = \begin{cases} 0 & \text{if } x_{k+i}^0 = 0 \\ 0 & \text{if } x_{k+i}^0 = 2 \\ 2 & \text{if } x_{k+i}^0 = 1 \end{cases}
\]

If \( x_{k+i}^0 = 1 \) find the first digit

in the sequence \( x_{k+2}^0, x_{k+3}^0, \ldots \) that is
different from 1. Then

\[
x_{k+i+1}^0, x_{k+i+2}^0, \ldots x_{k+i+n}^0 = \begin{cases} 2 & \text{if } x_{k+i}^0 = 0 \\ 0 & \text{if } x_{k+i}^0 = 2 \end{cases}
\]

and \( x_{k+n+1}^0, x_{k+n+2}^0, \ldots \) are chosen equal
to \( x_{k+n+i}^0, x_{k+n+i+1}^0 \ldots \).
Otherwise i.e., all \( x_{k+1}^0, x_{k+2}^0, \ldots \) are 1

Then choose \( x_{k+1}, x_{k+2}, \ldots \), all 2!

Now we have:

\[
| x - x_0 | = \left| (x_{k+1} - x_0^0) \frac{1}{3} + (x_{k+2} - x_0^0) \frac{1}{3^2} \right| + \left( x_{k+3} - x_0^0 \right) \frac{1}{3^3} + \cdots
\]

\[
\leq \frac{2}{3} \left( 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots \right) = \frac{2}{3} \cdot \frac{1}{1 - \frac{1}{3}} = \frac{1}{3} < \varepsilon
\]

and

\[
| f^k(x) - f^k(x_0) | = \left| (x_{k+1} - x_0^0) \frac{1}{3} \right| + \left( x_{k+2} - x_0^0 \right) \frac{1}{3^2} + \left( x_{k+3} - x_0^0 \right) \frac{1}{3^3} + \cdots
\]

\[
\geq \frac{2}{3}
\]

if \( x_{k+n} \not\in \{0, 1\} \)

\[
\begin{cases} 
\frac{1}{3^n} + \frac{1}{3^{n+1}} + \frac{1}{3^{n+2}} + \cdots & \text{if } x_0^0, x_0^0 \neq 1 \text{ or } 1 \text{ and } x_{k+n} \neq 1 \\
\frac{1}{3^n} + \frac{1}{3^{n+1}} \cdots + \frac{2}{3^{n+1}} & \text{if } x_0^0, x_0^0 \neq 1 \text{ and } x_{k+n} \neq 1
\end{cases}
\]

all \( n \) even

\[
| f^k(x) - f^k(x_0) | \geq \min \left\{ \frac{2}{3}, \frac{1}{2} + \frac{1}{2 \cdot 3^n}, \frac{1}{2} \right\}
\]

\[
\geq \frac{1}{2}
\]
Proposition: For \( g_4 : [0, 1] \rightarrow [0, 1] \)

\[ g_4(x) = 4x(1-x) \]

and any \( x_0 \in [0, 1] \), \( \varepsilon > 0 \) \( \exists \delta > 0 \) such that

\[ |x - x_0| < \delta \text{ and } |g_4(x) - g_4(x_0)| \geq \frac{\sqrt{2}}{4} \]

Proof: We will use continuity (or coding). Instead of understanding the orbit \( x, g_4(x), g_4^2(x), \ldots \) of a point we will only care if at each step it is in the left \( L = [0, \frac{1}{2}] \) or right \( R = [\frac{1}{2}, 1] \) sub-interval.

Hence the orbit of 0: 0, 0, 0, 0... will be \( LL\ldots \); the orbit of \( \frac{3}{4} : \)

\[ \frac{3}{4}, \frac{3}{4}, \frac{3}{4} \ldots \] will be \( RRR\ldots \); the orbit of \( \frac{1}{2} : \frac{1}{2}, 1, 0, 0, 0, \ldots \) will be \( LRLRL\ldots \) or \( RRLRL\ldots \).
To establish which parts have a certain timeliness note:

1) \[ g_4^L(L) = 0, \quad g_4^R(L) = 0 \]

More precisely:

\[ g_4^L(0, a_{11}) = L \]
\[ g_4^L(\{ 0, a_{11}, \frac{1}{2} \}) = R \]
\[ g_4^L(\{ \frac{1}{2}, a_{12} \}) = R \]
\[ g_4^L(\{ a_{12}, 1 \}) = L \]

where \[ 0 < a_{11} < \frac{1}{2} < a_{12} < 1 \] are such that

\[ g_4^L(a_{11}) = \frac{1}{2} = g_4^L(a_{12}) \quad i.e. \]

\[ a_{11} = \frac{1}{2} - \frac{\sqrt{2}}{4}, \quad a_{12} = \frac{1}{2} + \frac{\sqrt{2}}{4} \]
So the initial interval $I$ splits into $LL, LR$ at $a_1$ which means an initial point $x_0 \in LL \ i.e. \ x_0 \in [0, a_{11}]$
will have orbit $LL S_2 S_4... \ S_2 \in [L, R]$
while an initial point $x_0 \in LR \ i.e. \ x_0 \in [a_{11}, \frac{1}{2}]$
will have orbit $LR S_3...$

The $LL$ and $LR$ further split into subintervals $LLL, LRR$ at $a_{21}$

\[ g_4(a_{21}) = a_{11} \]

respectively $LRR, LRL$ at $a_{22}$

\[ g_4(a_{22}) = a_{12} \]

Similarly the initial interval $RR$ splits into $RR, RRL$ at $a_{12}$ into subintervals

$RRL, RRR$ at $a_{23}$ : $g_4(a_{23}) = a_{12}$

respectively

$RRL$, $RLL$ at $a_{24}$ : $g_4(a_{24}) = a_{11}$

The process repeats, each of the above subintervals splits again into 2 subintervals.

Details of the 8 subintervals are...
Conclusions

1) For any choice \( S_1, S_2, \ldots, S_k \in \mathbb{R} \)
there is an interval such that any point
in that interval has itinerary \( S_1, S_2, \ldots, S_k \).
We will see later that the length
of the interval is at most \( \frac{1}{2^{k+1}} \).

2) The length of \( LR \) and \( RK \)
intersection is \( \frac{\sqrt{2}}{4} \).

To finish the proof: Fix \( r_0 \in \mathbb{Q} \oplus \mathbb{Q} \)
and \( \varepsilon > 0 \). Consider \( LR \) such that
\( \frac{1}{2^{k+1}} < \varepsilon \). Let
\( S_1, S_2, \ldots, S_k \) be the itinerary
for \( x_0 \). Choose \( x \in \mathbb{Q} \oplus \mathbb{Q} \) in the
following way:
If $S_{n+1} S_{n+2} = LL$ or LR then $X \in S_1, S_2, \ldots, S_{n} LL$ or LR subsetwise.

If $S_{n+1} S_{n+2} = RR$ or RL then $X \in S_1, S_2, \ldots, S_{n} RR$ or RL subsetwise.

Consequently,

$$|X - x_0| < \frac{\nu}{2 \kappa} \leq \varepsilon$$

since they are both in the $S_1, S_2, \ldots, S_n$ subsetwise.

$$|g^L_k(x) - g^L_k(x_0)| \geq \frac{V^2}{4}$$

since, in the first case, $g^L_k(x) \in \text{RL}$ and $g^L_k(x_0) \in \text{LL}$ or LR which means they are separated by the RR subsetwise, which has length at least $\frac{\sqrt{2}}{4}$.

Similarly, in the second case, $g^L_k(x)$ and $g^L_k(x_0)$ are separated by LR which has the same length! Q.E.D.

Remark: The result that $g_k$ has points with sensitive dependence on initial condition is particularly
core of:

**Theorem (Li & Yorke '75)** If $S \subseteq \mathbb{R}$ is an interval and $f : S \to S$ is continuous and has a 3-periodic orbit, then there is an uncountable set of points in $S$ which have sensitive dependence on data!

Proof left as a final project, see Challenge 1 in the textbook.