§ 1.3 Stability of Fixed Points

Define consider the dynamical system

\[ f : S \rightarrow S \] and the fixed point

\[ p \in S : f(p) = p. \]

If for every neighborhood \( N_r(p) \) of \( p \)

there exists a neighborhood \( N_{r'}(p) \) of \( p \)

such that for all \( x_0 \in N_{r'}(p) \cap S \)

\[ f^n(x_0) = f \circ f \circ \ldots \circ f(x_0) \in N_r(p) \quad \forall n \in \mathbb{N} \]

for all \( n \in \mathbb{N} \)

then \( p \) is called a stable fixed point. If the fixed

point is not stable then it is called

unstable i.e. if there is a neighborhood

\( N_{r''}(p) \) of \( p \) such that for each neighborhood

\( N_{r'}(p) \) of \( p \) there is at least one point

\[ x \in N_{r'}(p) \cap S \] and \( n \in \mathbb{N} \) such that

\[ f^n(x) \notin N_{r''}(p). \]

then \( p \) is called unstable.
Remark In metric spaces \((S, d)\) the neighborhoods are defined by: \(\epsilon > 0\)

\[
N_\epsilon(p) = \{ x \in S \mid d(x, p) < \epsilon \}
\]

In particular if \(S \subseteq \mathbb{R}\) then

\[
N_\epsilon(p) = \{ x \in S \mid |x - p| < \epsilon \}
\]

while for \(S \subseteq \mathbb{R}^n\)

\[
N_\epsilon(p) = \{ x \in S \mid \|x - p\| < \epsilon \}
\]

where \(\|x - p\|\) is the Euclidean distance between \(x\) and \(p\). Later this distance will be denoted by \(|x - p|\).

Def (Sink) If \(p\) is a fixed point for \(f : S \to S\) and

\[
\exists N_\epsilon(p) \text{ such that } \forall x_0 \in N_\epsilon(p)
\]

\[
\lim_{n \to \infty} f^n(x_0) = p
\]

then \(p\) is called a sink or an attractive fixed point.
Def (source) If \( p \) is a fixed point for \( f : S \to S \) and

\[
\exists \epsilon > 0 \text{ such that for all } x_0 \in N_\epsilon(p) \ni x_0 \neq p \text{ we have } f^n(x_0) \notin N_\epsilon(p) \text{ for some } n \in \mathbb{N}
\]

then \( p \) is called a source or a repelling fixed point.

**Theorem** If \( S \subseteq \mathbb{R} \) and \( p \) is a fixed point for \( f : S \to S \) at which \( f \) is differentiable i.e., \( f'(p) \) exists, then

\[
|f'(p)| < 1 \text{ implies } p \text{ is stable and a sink.}
\]

while \( |f'(p)| > 1 \) implies \( p \) is unstable and a source.

**Proof** If \( |f'(p)| < 1 \) then

\[-1 < f'(p) < 1\]

and we can choose \( \delta > 0 \) such that

\[
f'(p) + \delta < 1 \text{ and } f'(p) - \delta > -1
\]

(e.g. \( \delta = \frac{1}{2} \max \{ 1 - f'(p), f'(p) + 1 \} \))
Since \( \lim_{x \to \nu} \frac{f(x) - f(\nu)}{x - \nu} = f'(\nu) \)

there is \( \varepsilon > 0 \) such that

\[
f'(\nu) - \frac{\varepsilon}{2} < \frac{f(x) - f(\nu)}{x - \nu} < f'(\nu) + \frac{\varepsilon}{2}
\]

for all \( x \in \left( \nu - \varepsilon, \nu + \varepsilon \right) \cap \left( 5, 10 \gamma \right) \)

\[
\Rightarrow \quad \left| \frac{f(x) - f(\nu)}{x - \nu} \right| < \alpha < 1
\]

where \( \alpha = \max \left\{ \left| f'(\nu) + \frac{\varepsilon}{2} \right|, \left| f'(\nu) - \frac{\varepsilon}{2} \right| \right\} \)

\[
\leq 1
\]

(2) \( \Rightarrow \quad |f(x) - \nu| \leq \alpha |x - \nu| \ \forall x \in N_{\varepsilon}(\nu)
\]

where we used \( f(\nu) = \nu \).

Then for any \( 0 < \delta \leq \varepsilon \) we get

\( x \in N_{\delta}(\nu) \Rightarrow f(x) \in N_{\delta}(\nu) \)

\( \Rightarrow \quad f^n(x) \in N_{\delta}(\nu) \) for all \( n \)

for \( 0 \leq \delta \leq \varepsilon \) we still have

\( x \in N_{\varepsilon}(\nu) \Rightarrow f(x) \in N_{\varepsilon}(\nu) \subset N_{\delta}(\nu) \)

\( \Rightarrow \quad f^n(x) \in N_{\varepsilon}(\nu) \subset N_{\delta}(\nu) \)
So \( p \) is stable. Moreover, from \( x \in N_{\varepsilon}(p) \cap S \) we have \( f(x), f^2(x), \ldots \) are all in \( N_{\varepsilon}(p) \cap S \) and by (6)

\[
\begin{align*}
|f^n(x) - p| &= |f(f^{n-1}(x)) - p| \\
&\leq \alpha |f^{n-1}(x) - p| \\
&\leq \ldots \leq \alpha^n |x - p|
\end{align*}
\]

Thus \( a < 1 \Rightarrow a^n \to 0 \) hence

\[
\lim_{n \to \infty} f^n(x) = p.
\]

If \( |f'(p)| > 1 \) we can choose \( \tilde{\delta} > 0 \) such that

\[
1 < f'(p) - \tilde{\delta}
\]

or \( -1 > f'(p) + \tilde{\delta} \)

and \( \tilde{\varepsilon} > 0 \) such that

\[
\tilde{f}'(p) - \tilde{\delta} < \frac{f(x) - f(p)}{x - p} < \tilde{f}'(p) + \tilde{\delta}
\]

for all \( x \in (p - \tilde{\varepsilon}, p + \tilde{\varepsilon}) \cap S \setminus \{p\} \).
In both cases we have
\[ \frac{|f(x) - f(y)|}{|x - y|} \to a > 1 \]

\[ \Rightarrow \quad |f(x) - y| > a|x - y| \]

for all \( x \in N_{\delta}(y) \setminus \{y\} \).

To prove \( p \) is a source we choose \( \varepsilon = \delta \). By contradiction,
if we assume \( \exists x_0 \in N_{\varepsilon}(p) \) such that
\[ \{x_0, f(x_0), \ldots, f^m(x_0), \ldots, y \} \subseteq N_{\varepsilon}(p) \]
then for all \( n \in \mathbb{N} \), \( f^n(x_0) \neq y \) and
\[ \varepsilon > |f^n(x_0) - y| > a^n|x_0 - y| \]

which contradicts \( a^n \to \infty \) since \( a > 1 \).

Remark: In general any source is an unstable fixed point. Indeed,
choose \( N_{\varepsilon}(p) \) in the definition of unstable to coincide with \( N_{\delta}(p) \) in the
Definition of a source. Now any neighborhood of \( p \) contains at least one point in \( N_{\varepsilon}(p) \nabla \{ p \} \). By the definition of a source, the orbit of this point eventually leaves \( N_{\varepsilon}(p) = N_{\delta}(p) \).

The theorem is now finished. Q.E.D.

**Examples**

1. \( f(x) = 2x \quad f(0) = 0 \quad f'(0) = 2 \)  
   => \( 0 \) is unstable and a source.

2. \( g(x) = 2x(1-x) \)
   \[ g(0) = 0 \quad g'(0) = 2 \]
   => \( 0 \) is unstable and a source
   \[ g\left(\frac{1}{2}\right) = \frac{1}{2} \quad g'(\frac{1}{2}) = 0 \]
   => \( \frac{1}{2} \) is stable and a sink

3. \( h(x) = x - x^3 \)
   \[ h(0) = 0 \quad h'(0) = 1 \]
   The theorem cannot be applied! However, \( 0 \) is stable and a sink, see HW 1