Continuous Dynamical Systems.

Differential Equations

Def: In continuous dynamical systems, the rate of change of the current state is given as a function of the current state. So if

\[ x(t) = (x_1(t), \ldots, x_n(t)) \] is the current state, then

\[ \frac{dx}{dt} = \left( \frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt} \right) \]

\[ = f(t, x(t)) \]

\[ = (f_1(t, x_1, \ldots, x_n), \ldots, f_n(t, x_1, x_2, \ldots, x_n)) \]

which is an ordinary differential system of equations of order 1, i.e., the unknown functions depend on only one variable (time) and the highest derivative appearing in the equation is the first derivative (derivative must appear...
Otherwise it will not be called "differentiable\). \)

Remark. All higher order equations can be reduced to first order. For example, in vertical motion under gravity:\)

\[ \dot{z} = g(z) \]

where \( g(z) = -g \) when the gravitational field is assumed constant or:

\[ g(z) = -\frac{MG}{(R+z)^2} \]

where \( M = \) mass of the earth

\( G = \) gravitational constant

\( R = \) radius of the earth.

In any case the state of the system will be given by position \( z \) and velocity \( \dot{z} \):

\[ x = (x_1, x_2) = (z, \dot{z}) \]

Hence:

\[
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = g(x_1)
\end{cases}
\]
where \( f(x_1, x_2) = (x_2, g(x_1)) \)

More generally

\[
y^{(n)} = g(t, y, y', \ldots, y^{(n-1)})
\]

\[
x = (x_1, \ldots, x_n) = (y, y', \ldots, y^{(n-1)})
\]

\[
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = x_3 \\
\vdots \\
\dot{x}_{n-1} = g(t, x_1, x_2, \ldots, x_n)
\end{cases}
\Rightarrow \dot{x} = f(t, x)
\]

Same argument can be applied to higher order systems of equations.

**Def.** An **autonomous differential eq** (dynamical system) does not depend explicitly on the independent variable (time):

\[
\dot{x} = f(x)
\]

**Remark.** Non-autonomous differential equations

\[
\dot{x} = f(t, x)
\]
can be transformed in autonomous ones via:

\[ y = (y_1, y_2, ..., y_{n+1}) = (t, x_1, ..., x_n) = (t, x) \]

\[ y' = g(y) \text{ where } y(t) = (t, f(y)), \quad i.e. \]

\[ \begin{cases} 
  \dot{y}_1 = 1 \\
  (\dot{y}_2, ..., \dot{y}_n) = \dot{x} = f(t,x) = f(y)
\end{cases} \]

Def (solutions) Consider

\[ \dot{x} = f(t,x) \]

\[ f: \mathbb{R} \to \mathbb{R}^n \quad \mathbb{U}_f \subseteq \mathbb{R} \times \mathbb{R}^n \text{ open} \]

a function \( x: J \to \mathbb{R}^n \) where \( J \subseteq \mathbb{R} \)

is an \underline{integral} is a solution if:

\( x \) is differentiable on \( J \).

\( (t, x(t)) \in \mathbb{U}_f \quad \forall t \in J \)

and:

\[ \dot{x}(t) = f(t, x(t)) \quad \forall t \in J. \]
Important properties of systems of differential equations

**Theorem (Local existence & uniqueness)**

Consider

\[ \dot{x} = f(t, x) \]

where \( f: \mathbb{R} \rightarrow \mathbb{R}^n \) \( \mathbb{R} \subseteq \mathbb{R}^n \) open is continuous in a neighborhood \( N \subseteq \mathbb{R} \)

of a point \((t_0, x^0)\). Then the equation has at least one solution passing through \((t_0, x^0)\) i.e., \( x(t_0) = x^0 \).

Moreover, if \( f \) is Lipschitz w.r.t. \( x \)

in \( N \) i.e., \( \exists L > 0 \) such that

\[ \| f(t, x_1) - f(t, x_2) \| \leq L \| x_1 - x_2 \| \]

\( \forall (t, x_1, t, x_2) \in N \).

Then for any two solutions \( x, \tilde{x} \)

with \( x(t_0) = x^0 = \tilde{x}(t_0) \) \( \exists J \in t_0 \)

an interval such that

\[ x(t) = \tilde{x}(t) \quad \forall t \in J. \]
Sketch of proof

\begin{align*}
(1) \quad \begin{cases} 
\dot{x} &= f(t; x) \\
 x(t_0) &= x^0
\end{cases} \\
(2) \quad x(t) &= x^0 + \int_{t_0}^{t} f(s; x(s)) \, ds 
\end{align*}

i.e., any sln of (1) is a continuous sln of (2) and any cont sln of (2) is a sln of (1).

Let \( N_{\varepsilon, \nu} = [b_0 - \varepsilon, b_0 + \varepsilon] \times B(x^0, \nu) \subseteq \mathbb{R}^n \)
where \( B(x^0, \nu) = \{ y \in \mathbb{R}^n \mid \| y - x^0 \| \leq \nu \} \)
\( N_{\varepsilon, \nu} \) is compact

\[ f \text{ continuous} \implies \exists M > 0 : \| f(t, y) \| \leq M \quad \forall (t, y) \in N_{\varepsilon, \nu} \]

Let \( \delta \leq \varepsilon : M\delta \leq \nu \)

\[ C_{\delta} = \{ y : [b_0 - \delta, b_0 + \delta] \to \mathbb{R}^n \mid y \text{ continuous} \} \]
\[ \|y\| = \sup_{t \in [-\delta, \delta]} \|y(t)\| \]

\[ y_0(t) = x^0, \quad t \in [b_0 - \delta, b_0 + \delta] \]

\[ C_\delta, \nu = \{ y \in C_\delta \mid \|y - y_0\|_{\infty} \leq \nu \} \]

\[ S : C_\delta, \nu \rightarrow C_\delta, \nu \]

\[ (Sy)(t) = x^0 + \int_{b_0}^{t} f(s, y(s)) \, ds. \]

Moreover, for

\[ K = \{ y \in C_\delta, \nu \mid \forall t_1, t_2 \in [b_0 - \delta, b_0 + \delta] \]

\[ \|y(t_1) - y(t_2)\| \leq M |t_1 - t_2| \} \]

we have \( S(K) \subseteq K \ (\text{Shauder Theorem}) \)

\( K \) convex and compact

\( S \) continuous

\( S \) has a fixed point.

But for Lipschitz \( f \) the proof is simpler. Choose \( \delta \) maybe smaller.
such that $\delta L < 1$. Then

$$S : C_0, r \rightarrow C_{\delta, r}$$

is a contraction

$$\| S(y_1) - S(y_2) \|_\infty \leq \delta L \| y_1 - y_2 \|_\infty$$

Consider the sequence

$$y_1 = S y_0$$

$$y_2 = S y_1$$

$$\vdots$$

$$y_k = S y_{k-1}$$

$$\vdots$$

The sequence is Cauchy (fundamental)

Hence convergent $y_k \rightarrow x$

$S$ continuous $\Rightarrow S y_k \rightarrow S x$

But $y_{k+1} = S y_k$ $\Rightarrow x = S x$ $\Rightarrow$ e.d.