4. Chaos in Higher Dimensions

Lyapunov Exponents

**Def.** Let \( \mathbf{S} \subseteq \mathbb{R}^m \), \( f : \mathbf{S} \rightarrow \mathbf{S} \), \( \{ x_0, f(x_0) = x_1, f^2(x_0) = x_2, \ldots \} \subseteq \mathbf{S} \) an orbit. Assume \( \mathbf{D} f^u(x_0), \mathbf{u} \in \mathbb{R}^m \), exist.

For each \( \mathbf{u} \in \mathbb{R}^m \) let \( \lambda_1^u \geq \lambda_2^u \geq \ldots \geq \lambda_m^u \geq 0 \) be the \( \lambda \)-values of \( \mathbf{D} f^u(x_0) [\mathbf{D} f^u(x_0)]^T \) and \( \lambda_i^u = \sqrt{\lambda_i^u} \). Then

\[
L_i = \lim_{n \to \infty} \left( \lambda_i^u \right)^{\frac{1}{n}}
\]

is the \( i \)th Lyapunov number of the orbit \( \mathbf{x}_0 \). Note that

\[
L_1 \geq L_2 \geq \ldots \geq L_m \geq 0
\]

The corresponding Lyapunov exponents are

\[
\lambda_i := \ln L_i \quad i = 1, m.
\]

Remark 1: \( \lambda_i, i = 1, m \) are also the critical values of the map:

\[
\mathbf{v} \rightarrow \| \mathbf{D} f^u(x_0) \mathbf{v} \|^2 \quad \mathbf{v} \in \mathbb{R}^m, \| \mathbf{v} \| = 1.
\]
They can also be characterized by:

\[ \rho_i^m = \min \max_{\|v\| = 1} \| Df^u(x_0)v \| \]

Hence

\[ \rho_1^m = \max_{\|v\|=1} \| Df^u(x_0)v \| \]

is the maximal dilation of a vector by the linear map \( Df(x_0) \),

\[ \rho_2^m = \min_{v_i \in \mathbb{R}^m, v_i \perp v_j, \|v_i\|=1} \max_{\|v\|=1} \| Df^u(x_0)v \| \]

is the maximal dilation in the subspace orthogonal to the direction in which the actual maximal dilation occurs, etc.

\[ \rho_m^m = \min_{v_i \in \mathbb{R}^m, v_i \perp v_j, \|v_i\|=1} \max_{\|v\|=1} \| Df^u(x_0)v \| \]

\[ = \min_{\|v\|=1} \| Df^u(x_0)v \| \]

is the minimal possible dilation.

Remark 2: There are other definitions of Lyapunov exponents. For example
for each $v \in \mathbb{R}^m$ the upper Lyapunov exponent:

$$l(v) = \limsup_{n \to \infty} \frac{1}{n} \ln \|A^n f(x_0) v\|$$

becomes the Lyapunov exponent in direction $v$ of the limit instead of limsup exists.

Based on:

$$\{ l(\lambda v) = l(v) : \forall \lambda \in \mathbb{R} \setminus \{0\} \}$$

(1)

$$\max \left\{ l(v_1), l(v_2) \right\} \leq l(y + v_1 + v_2) \leq \max \left\{ l(v_1), l(v_2) \right\}$$

and

$$l(v_1) < l(v_2) \implies l(v_1 + v_2) = l(v_2)$$

one can show that there are an integer $k \leq m$ the numbers: $l_1 < l_2 < \ldots < l_k$ and the subspaces $\{0\} \subset E_1 \subset E_2 \subset \ldots \subset E_k = \mathbb{R}^m$

such that

$$l(v) \leq l_i \forall v \in E_i$$

and

$$l(v) = l_i \forall v \in E_i \setminus E_{i-1}.$$
can play the role of the Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$.

In fact, Oseledec Theorem proves that if $f$ admits an invariant probability measure, then the two definitions coincide for all $x_0 \in S$ except maybe a set of measure zero.

\underline{Particular case:}

If $x_0 = f(x_0)$, i.e. $x_0$ is a fixed point, then

$$Df^u(x_0) = [Df(x_0)]^u$$

and it is easy to see using the Jordan normal form of $Df(x_0)$ in Lemma 8) that

$$Q(v) \in \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \quad \forall \ v \in \mathbb{R}^n$$

where $\lambda_1, \ldots, \lambda_k$ are the $e$-values of $Df(x_0)$ with $|\lambda_1| < |\lambda_2| < \ldots < |\lambda_k|$. Hence

$$\lambda_i = |\lambda_i| \quad i = 1, k \quad \text{and}$$

$$E_i = E_{i-1} \oplus V_i \quad i = 1, k$$

where
$\mathcal{V}_i = \text{subspace generated by all (generalized) } \mathcal{C}\text{-vectors of all } \mathcal{C}\text{-values with absolute value equal to } |\mathcal{C}_i|$. 