§ 4.5 Box Dimension

In what follows a box of size \( \varepsilon > 0 \) in \( \mathbb{R}^n \) is the set:

\[
[\mathbf{x}_1, \mathbf{x}_1 + \varepsilon] \times [\mathbf{x}_2, \mathbf{x}_2 + \varepsilon] \times \cdots \times [\mathbf{x}_n, \mathbf{x}_n + \varepsilon]
\]

i.e. an \( n \)-cube with vertex at \((\mathbf{x}_1, \ldots, \mathbf{x}_n)\) and sides parallel to the axes.

A grid of size \( \varepsilon > 0 \) made of such boxes with vertices at integer multiples of \( \varepsilon \):

\[
(2\mathbf{e}, 2\mathbf{e}, \ldots, 2\mathbf{e}) \quad 2, 2, \ldots, 2 \in \mathbb{Z}
\]

**Def.** If \( S \subseteq \mathbb{R}^n \) bounded the box dimension of \( S \) is defined as:

\[
\text{boxdim } S = \lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)}
\]

if the limit exists. Here

\[
N(\varepsilon) = \text{# of boxes in the } \varepsilon \text{-grid the set } S \text{ intersects}.
\]

**Properties**

1. If \( \{b_n\}_{n \geq 1} \subseteq \mathbb{R} \) satisfies \( b_1 > b_2 > \cdots > b_n > b_{n+1} > \cdots \), \( \lim_{n \to \infty} b_n = 0 \) and
\[
\lim_{u \to \infty} \frac{\ln b_{u+1}}{\ln b_u} = 1 \text{ then } \forall S \subseteq \mathbb{R}^n
\]

S converged we have:

\[
\text{boxdim } S = \lim_{u \to \infty} \frac{\ln \left( N_u(b_u) \right)}{\ln \left( \frac{1}{b_u} \right)}
\]

provided the limit exists.

2° If

\[ N_0(\varepsilon) = \text{ minimum # of } \varepsilon \text{ boxes} \]

needed to cover \( S \),

then \( \forall S \subseteq \mathbb{R}^n \) bounded

\[
\text{boxdim } S = \lim_{\varepsilon \to 0} \frac{\ln N_0(\varepsilon)}{\ln \left( \frac{1}{\varepsilon} \right)}
\]

3° Boxes in the def of boxdim

can be replaced by other geometrical objects

of size \( \varepsilon \) such that:

(i) any \( \varepsilon \)-box can be covered by a fixed # of \( \varepsilon \)-sized objects

(ii) any \( \varepsilon \)-sized object can be covered by a fixed # of \( \varepsilon \)-boxes.

Example includes: balls of radius \( \varepsilon \), similar triangles of one side of length \( \varepsilon \) in \( \mathbb{R}^2 \), etc.
Proof.  

1. \( A \leq \epsilon_2 \implies \) 

\[
\frac{b_{n+1}}{b_n} \leq \epsilon < \frac{b_n}{b_{n-1}}
\]

Now any \( \epsilon \) box is covered by \( 2^m \)

\( b_n \) boxes. Similarly, any \( b_{n+1} \) box is

covered by \( 2^m \) \( \epsilon \)-boxes, hence

\[
2^{-m} N(b_n) \leq N(\epsilon) \leq 2^{-m} N(b_{n+1})
\]

Then for \( n \geq n_0 \) such that \( b_{n+1} < \frac{1}{\epsilon} \)

we have

\[
\ln \frac{1}{b_n} < \ln \frac{1}{\epsilon} \leq \ln \frac{1}{b_{n+1}}
\]

and \( \forall n \geq n_0 \) and \( b_n \leq \epsilon \leq b_{n+1} \) :

\[
\ln \left( \frac{2^{-m} N(b_n)}{\ln \left( \frac{1}{b_{n+1}} \right)} \right) \leq \ln \frac{N(\epsilon)}{\ln \left( \frac{1}{\epsilon} \right)} \leq \ln \left( \frac{2^{-m} N(b_{n+1})}{\ln \left( \frac{1}{b_n} \right)} \right)
\]

\[
= \frac{-m \ln 2}{\ln \left( \frac{1}{b_{n+1}} \right)} + \ln \left( N(b_n) \right) \leq \ln \frac{\ln \left( \frac{1}{\epsilon} \right)}{\ln \left( \frac{1}{b_n} \right)} + \ln \frac{\ln \left( \frac{1}{\epsilon} \right)}{\ln \left( \frac{1}{b_{n+1}} \right)}
\]

\[
\leq \ln \frac{N(\epsilon)}{\ln \left( \frac{1}{\epsilon} \right)} \leq \frac{\ln \left( \frac{1}{\epsilon} \right)}{\ln \left( \frac{1}{b_n} \right)} + \ln \frac{\ln \left( \frac{1}{\epsilon} \right)}{\ln \left( \frac{1}{b_{n+1}} \right)}
\]

If \( \lim_{n \to \infty} \frac{\ln N(b_n)}{\ln \left( \frac{1}{b_n} \right)} = \infty \)

\[
\lim_{n \to \infty} \frac{\ln \left( \frac{1}{b_n} \right)}{\ln \left( \frac{1}{b_{n+1}} \right)} = 1
\]
and \( \lim_{n \to \infty} b_n = 0 \) \( \implies \lim_{n \to \infty} \ln \left( \frac{1}{b_n} \right) = +\infty \)

then both the left most side and the right most side of the above ineq converge to \( \lim_{n \to \infty} \frac{\ln N(\frac{1}{b_n})}{\ln \left( \frac{1}{b_n} \right)} \) leading:

\[ \lim_{\varepsilon > 0} \frac{\ln N(\frac{1}{\varepsilon})}{\ln \left( \frac{1}{\varepsilon} \right)} \rightarrow \lim_{n \to \infty} \frac{\ln N(\frac{1}{b_n})}{\ln \left( \frac{1}{b_n} \right)} \] \text{ Q.E.D.} \]

2° Similar proof using:

\[ \frac{1}{2^m} N(\varepsilon) \leq N_0(\varepsilon) \leq N(\varepsilon) \]

since any \( \varepsilon \)-box cannot be covered by at most \( 2^m \) \( \varepsilon \)-boxes in an \( \varepsilon \)-grid.

3° If \( N_1(\varepsilon) = \# \text{ of } \varepsilon \text{-sized objects}

\text{intersecting } S. \]

then

\[ \frac{1}{u_1} N(\varepsilon) \leq N_1(\varepsilon) \leq u_2 N(\varepsilon) \]

where \( u_1, u_2 \) are the numbers given in (i) and (ii). The proof proceeds as in 1°.
Applications

Ex 1 The contour set $K_\infty$:

$$\text{box dim } K_\infty = \frac{\ln 2}{\ln 3}.$$  

Indeed, let $b_n = \frac{1}{3^n}$ then

$$N_0(b_n) = 2^n$$  

because $K_n \subsetnic K_\infty$, see previous lecture, is made of $2^n$ intervals of length $\frac{1}{3^n}$ and their endpoints remain in $K_\infty$. Using $1^\circ$  

$$\Rightarrow \text{box dim } K_\infty = \lim_{n \to \infty} \frac{\ln (2^n)}{\ln (3^n)^{\frac{1}{\ln 3}} = \frac{\ln 2}{\ln 3}}$$

Ex 2 The Sierpinski Carpet's limiting set

$$\text{(attache) } = K_\infty \times [0,1]^d$$

$$\text{box dim } (K_\infty \times [0,1]^d) = \text{box dim } K_\infty + \text{box dim } [0,1]^d$$

$$= \frac{\ln 2}{\ln 3} + 1$$

Indeed, with $b_n = \frac{1}{3^n}$ we now need

$$N_0(b_n) = 3^n 2^n$$  

squares of size $\frac{1}{3^n}$

to cover $K_\infty \times [0,1]^d$.  


Ex 3. Sierpinski Carpet: \( K_\infty \times K_\infty \)

\[
\text{boxdim} \ K_\infty \times K_\infty = \frac{\ln 4}{\ln 3}
\]

\[
= \text{boxdim} \ K_\infty + \text{boxdim} \ K_\infty
\]

Indeed, now we need \( 2^n \cdot 2^n = 4^n \) squares of size \( \frac{1}{3^n} \) to cover \( K_\infty \times K_\infty \).

Property 4°. If \( A \subseteq \mathbb{R}^{n_1} \), \( B \subseteq \mathbb{R}^{n_2} \) have box dimensions \( d \) and \( e \), then:

\[
A \times B \subseteq \mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1+n_2} \quad \text{has box dimension} \quad d + e.
\]

Proof. See HW.

Ex 4. Sierpinski Gasket SG:

\[
\text{boxdim}(SG) = \frac{\ln 3}{\ln 2}
\]

Indeed, we use triangles similar to the initial one but of horizontal side of length...
where \( L \) is the length of the bottom side of the initial triangle.

So \( b_n = \frac{L}{2^n} \) and

\[
H_0(b_n) = \frac{3^n}{n} \]

\[
\Rightarrow \text{box dim} \ (S_0) = \lim_{n \to \infty} \frac{\ln 3^n}{\ln \frac{3^n}{n}} = \frac{\ln 3}{\ln 2}
\]

**Ex 5** Any curve \( \gamma : [0,1] \to \mathbb{R}^m \)

\( C^1 \) and \( 1 \frac{d\gamma(t)}{dt} \neq 0 \ \forall \ t \in (0,1) \)

\( \text{box dim} \ \gamma = 1 \)

Any surface \( S : [0,1] \times [0,1] \to \mathbb{R}^m \)

\( C^1 \) and such that the vectors \( \frac{\partial S}{\partial s}(s,t) \)

\( \frac{\partial S}{\partial t}(s,t) \) are linearly indep \( \forall (s,t) \in (0,1) \times (0,1) \)

\( \text{box dim} \ S = 2 \)

**Remark** Since curves have length and surfaces have area, then an integer box dimension sometimes predicts the volume size of the object i.e. whether it has length,
area, volume, etc. However for "bad" sets the prediction may be wrong as the following example shows.

Ex 6 \( Q_1 = Q \cap [0,1] \) set of rationals in \([0,1]\) uncountable. We have

\[
\text{borel} \text{dim } Q_1 = 1
\]

\[
\text{meas } (Q_1) = 0.
\]

Indeed, for \( b_n = \frac{1}{n} \)

\[
\text{b}(b_n) \in \mathbb{N} \text{ since there is at least one rational in } \left[ \frac{k}{n}, \frac{k+1}{n} \right], \quad k \in \{0,1,...,n-1\}
\]

\[
\Rightarrow \text{borel} \text{dim } Q_1 = \lim_{n \to \infty} \frac{\ln n}{\ln n} = 1.
\]

But if \( R_1, R_2, ... \) is an accumulation of rationals in \([0,1]\), then, \( A \in \mathbb{R} \)

\[
\bigcup_{n=1}^{\infty} \left[ \frac{R_n - \frac{\varepsilon}{2n}}{2}, \frac{R_n + \frac{\varepsilon}{2n}}{2} \right] \text{ is a cover of } A,
\]

with subsets of total length:

\[
\frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \cdots = \frac{\varepsilon}{2} \left( 1 + \frac{1}{2} + \frac{1}{4} + \cdots \right) = \frac{\varepsilon}{2} \left( 1 - \frac{1}{2} \right) = \frac{\varepsilon}{2}.
\]

\[
\Rightarrow \text{meas } (Q_1) = 0.
\]
Theorem: If $S \subseteq \mathbb{R}^m$ is bounded and $\text{Boxdim } S = d < m$, then $\text{mes } (S) = 0$.

Proof: By definition & property 2°

Existence of $\varepsilon_0 > 0$ such that

\[
\ln N_0(\varepsilon) \leq \frac{d + m}{2} \ln \frac{1}{\varepsilon} \quad \forall \varepsilon \leq \varepsilon_0
\]

\[
\Rightarrow \quad N_0(\varepsilon) \leq \left( \frac{1}{\varepsilon} \right)^\frac{d + m}{2} \quad \forall \varepsilon \leq \varepsilon_0
\]

For any $\delta > 0$ we can choose $\varepsilon \leq \varepsilon_0$ such that $\varepsilon^{\frac{m-d}{2}} \leq \delta$

and we have a cover of $S$ with $\varepsilon$-boxes (boxes) of total volume

\[
\text{Vol} = N_0(\varepsilon) \cdot \varepsilon^m \leq \varepsilon^{\frac{m-d}{2}} \leq \delta
\]

Since $\delta > 0$ is arbitrary $\Rightarrow \text{mes } S = 0$.