Generalities

**Def.** (Dynamical Systems) A set of states together with a rule to determine the current state from previous ones.

Example: $f: S \to S$. Here $S$ is the set of states and the current state $f(s_0)$ is determined from the previous one via the rule given by the function $f$!

Remark. If the current state depends on more than one (say 2) previous states we can still describe it as a function:

$$F: S^2 \to S = S \times S.$$  

$$F(s_1, s_2) = (s_2, f(s_1, s_2)).$$
Def (orbit) For the dynamical system \( f : S \to S \) the orbit of \( x_0 \in S \) is
\[
\{ x_0, f(x_0), f(f(x_0)), \ldots \}
\]

Def (fixed point) For the dynamical system \( f : S \to S \), \( p \in S \) is a fixed point if
\[
f(p) = p.
\]

One-dimensional maps

Example 1 \( f : \mathbb{R} \to \mathbb{R} \) \( f(x) = 2x \)
The orbit of \( x_0 \in S \) is
\[
\{ x_0, 2x_0, 2^2 x_0, \ldots, 2^n x_0, \ldots \}
\]
diverges to \( +\infty \) if \( x_0 > 0 \), \(-\infty \) if \( x_0 < 0 \).
\[
f(p) = kp \Rightarrow 2p = p \Leftrightarrow p = 0
\]
0 is the only fixed point.
Example 2 \[ g : [0,1] \to [0,1] \]
\[ g(x) = 2x(1-x) \]

There is no easy formula for the orbit, but:

- The orbit of 0 is \( \{ 0 \} \).
- The orbit of 1 is \( \{ 1, 0, 1, 0, \ldots \} \).
- The orbit of \( 0 < x_0 < 1 \) converges to \( \frac{1}{2} \) i.e.

\[ \lim_{n \to \infty} g^n(x_0) = \frac{1}{2} \]

Use \( g^n(x_0) = g \circ g \circ \ldots \circ g(x_0) \) \( n \) times

Proof:

\[ y = x \]
If $0 < x_0 < \frac{1}{2}$ then

$$x_0 < g(x_0) < \frac{1}{2} \quad \text{{(cyclic)}}$$

Intuitively we have

$$x_0 < g(x_0) < g^2(x_0) < \ldots < \frac{1}{2}$$

Hence

$$\{ g^n(x_0) \} \text{ is strictly increasing and bounded above by } \frac{1}{2}$$

$\Rightarrow \lim_{n \to \infty} g^n(x_0) = a \text{ exists}$

and $0 < x_0 < a < \frac{1}{2}$.

But $g^{n+1}(x_0) = g(g^n(x_0))$ $\xrightarrow{n \to \infty} \int_a^g f$ is continuous

$\Rightarrow a = g(a) \Rightarrow a = 2a(1-a)$

$\Rightarrow a \in \{ 0, \frac{1}{2} \}$

Hence the limit is $\frac{1}{2}$. 
If $\frac{1}{2} < x_0 < 1$ then

$0 < g(x_0) < \frac{1}{2}$ check!

and the argument repeats

If $x_0 = \frac{1}{2}$ $g(x_0) = \frac{1}{2}$ and the orbit is $\frac{1}{2}$.

Of course by now we have already discovered the fixed points:

$g(x) = x \in \{0, \frac{1}{2}, \frac{1}{4}\}$