1. The joint probability density function of $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} \frac{6}{7} \left( x^2 + \frac{xy}{2} \right) & 0 < x < 1, 0 < y < 2, \\ 0 & \text{otherwise} \end{cases}$$

(i) (10 points) Find $P(X < Y)$.

(ii) (15 points) Find $P \left( Y < \frac{1}{2} \mid X < \frac{1}{2} \right)$

**Solution:** These are two parts of Problem 5 in midterm 2. You can find the solution of midterm 2 in the course page.
2. A man and a woman agree to meet at a certain location about 12:30 P.M. If the man arrives at a time uniformly distributed between 12:15 and 12:45, and if the woman independently arrives at a time uniformly distributed between 12:00 and 1:00 P.M., find the probability that the first to arrive waits no longer than 5 minutes. What is the probability that the man arrives first?

Solution: We assume the man and the woman arrive at 12:X and 12:Y respectively (1:00 is 12:60). Then the space of X and Y are [15, 45] and [0, 60] respectively, and \( f_X(x) \equiv 1/(45 - 15) = 1/30, \) \( f_Y(y) \equiv 1/(60 - 0) = 1/60. \) So the joint pdf of X and Y is
\[
f(x, y) = \frac{1}{30} \cdot \frac{1}{60} = \frac{1}{1800}, \quad (x, y) \in [15, 45] \times [0, 60].
\]

As a result,
\[
P(|X - Y| \leq 5) = \int_{15}^{45} \int_{x-5}^{x+5} f(x, y) \, dy \, dx
\]
\[
= \frac{1}{1800} \int_{15}^{45} y|x+5|_x-5 \, dx
\]
\[
= \frac{30 \times 10}{1800}
\]
\[
= \frac{1}{6},
\]

and
\[
P(X < Y) = \int_{15}^{45} \int_{x}^{60} f(x, y) \, dy \, dx
\]
\[
= \frac{1}{1800} \int_{15}^{45} y|60|_x \, dx
\]
\[
= \frac{1}{1800} \int_{15}^{45} (60 - x) \, dx
\]
\[
= \frac{60x - x^2/2}{1800} \bigg|_15^{45}
\]
\[
= \frac{1}{2}.
\]