1. Sixty percent of students at a certain school wear neither a ring nor a necklace. Twenty percent wear a ring and 30 percent wear a necklace. If one of the students is chosen randomly, what is the probability that this student is wearing:

(a) a ring or a necklace?
(b) a ring and a necklace?

**Solution:** If \( R \) (\( N \), resp.) is the event of wearing a ring (necklace, resp.), then \( P(R) = 20\% \), \( P(N) = 30\% \) and \( P((R \cup N)^c) = 60\% \); hence

(a) the event of wearing “either a ring or a necklace” is \( R \cup N \). The probability of it is

\[
P(R \cup N) = 1 - P((R \cup N)^c) = 1 - 60\% = 40\%.
\]

(b) the event of wearing “a ring and a necklace” is \( R \cap N \). So the probability is

\[
P(R \cap N) = P(R) + P(N) - P(R \cup N) = 20\% + 30\% - 40\% = 10\%.
\]
2. The game of craps is played as follows: a player rolls two dice. If the sum is 2, 3, or 12, the player loses; if the sum is either a 7 or an 11, the player wins. If the outcome is anything else, the player continues to roll the dice until he rolls either the initial outcome or a 7. If the 7 comes first the player loses, whereas if the initial outcome reoccurs before the 7 appears, the player wins. Compute the probability of a player winning at craps.

Solution: If $E_i$ is the event that the initial outcome is $i$ and the player wins, then the probability of winning is $\sum_{i=2}^{12} P(E_i)$ since $E_i$’s are mutually exclusive.

To compute $P(E_i)$, we first consider the experiment of rolling two dices once. The sample space is $S = \{(i, j) \mid 1 \leq i, j \leq 6\}$ where every pair $(i, j)$ occurs equally likely. The following are the outcomes corresponding to sums of two dices:

- $2 : (1, 1)$
- $3 : (1, 2), (2, 1)$
- $4 : (1, 3), (2, 2), (3, 1)$
- $5 : (1, 4), (2, 3), (3, 2), (4, 1)$
- $6 : (1, 5), (2, 4), (3, 3), (4, 2), (5, 1)$
- $7 : (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$
- $8 : (2, 6), (3, 5), (4, 4), (5, 3), (6, 2)$
- $9 : (3, 6), (4, 5), (5, 4), (6, 3)$
- $10 : (4, 6), (5, 5), (6, 4)$
- $11 : (5, 6), (6, 5)$
- $12 : (6, 6)$

If $P_i$ is the number of outcomes summing to $i$, then $P_2 = P_{12} = 1, P_3 = P_{11} = 2, P_4 = P_{10} = 3, P_5 = P_9 = 4, P_6 = P_8 = 5$ and $P_7 = 6$ by counting the outcomes. (1 pt)

Now we consider two cases.

If $i \in \{2, 3, 7, 11, 12\}$, we know by the rule of game that $P(E_i) = 0$ for $i = 2, 3, 12$ and $P(E_i) = P_i/36$ for $i = 7, 11$. (1 pt)

If $i \in \{4, 5, 6, 8, 9, 10\}$, we define $E_{i,n}$ to be the event of winning after $n$ rolls when the first roll has sum $i$. This is an experiment with $36^n$
equally likely outcomes. The event $E_{i,n}$ happens when the first rolls and the last roll get any of $P_i$ outcomes summing to $i$, and the other $n-2$ rolls in between get any sum besides $7$ and $i$, which have $36 - P_i - P_7 = 30 - P_i$ choices. By the basic principle of counting, $E_{i,n}$ contains $P_i(30 - P_i)^{n-2}P_i = P_i^2(30 - P_i)$ distinct outcomes, hence the probability

$$P(E_{i,n}) = P_i^2(30 - P_i)^{n-2}/36^n. \quad (1 \text{ pt})$$

Since $E_{i,n}$ ($n = 2, 3, \ldots$) are mutually exclusive,

$$P(E_i) = P\left(\bigcup_{n=2}^{\infty} E_{i,n}\right)$$

$$= \sum_{n=2}^{\infty} P(E_{i,n})$$

$$= \sum_{n=2}^{\infty} \frac{P_i^2(30 - P_i)^{n-2}}{36^n}$$

$$= \frac{P_i^2}{36^2} \sum_{k=0}^{\infty} \left(\frac{30 - P_i}{36}\right)^k$$

$$= \frac{P_i^2}{36^2} \frac{1}{1 - (30 - P_i)/36}$$

$$= \frac{P_i^2}{36(6 + P_i)} \quad (1 \text{ pt})$$

for $i \in \{4, 5, 6, 8, 9, 10\}$.

Now by plugging in the values of $P_i$, we get the probability of a player winning at craps is

$$\frac{1}{6} + \frac{1}{18} + 2 \cdot \frac{1}{36} \left(\frac{3^2}{6 + 3} + \frac{4^2}{6 + 4} + \frac{5^2}{6 + 5}\right) = \frac{244}{295} \approx 0.493. \quad (1 \text{ pt})$$