Problem I (Hölder and Minkowski Inequalities) This exercise will help you prove two very important inequalities in analysis, the second one being essential in showing that \( d_p, 1 \leq p < \infty \) are metrics on \( \mathbb{R}^k \). We assume that we have already proven that the function \( f : [0, \infty) \to \mathbb{R}, f(x) = x^{1/p}, p \geq 1 \) is strictly increasing and concave:
\[
f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y), \quad \text{for all } 0 \leq \alpha \leq 1, x \geq 0, y \geq 0.
\]

(i) Fix \( k \in \mathbb{N} \), and consider the real numbers \( 0 \leq \alpha_n \leq 1, n = 1, 2, \ldots, k, \) such that \( \sum_{n=1}^{k} \alpha_n = 1 \). Show that for the function \( f \) defined above and any real numbers \( x_n > 0, n = 1, 2, \ldots, k \) we have:
\[
f\left( \sum_{n=1}^{k} \alpha_n x_n \right) \geq \sum_{n=1}^{k} \alpha_n f(x_n).
\]

(ii) Show Hölder Inequality: for any nonnegative real numbers: \( x_n \geq 0, y_n \geq 0, n = 1, 2, \ldots, k, \) and any \( p \geq 1, q \geq 1 \) such that \( 1/p + 1/q = 1 \) we have
\[
\sum_{n=1}^{k} x_n y_n \leq \left( \sum_{n=1}^{k} x_n^p \right)^{1/p} \left( \sum_{n=1}^{k} y_n^q \right)^{1/q}.
\]
(Hint: Use the inequality in part (i) with \( \alpha_n = y_n^q / \left( \sum_{n=1}^{k} y_n^q \right) \), and \( x_n \) replaced by \( x_n^p / (y_n^q) \).)

(iii) Show Minkowski Inequality: for any nonnegative real numbers: \( x_n \geq 0, y_n \geq 0, n = 1, 2, \ldots, k, \) and any \( p \geq 1 \) we have:
\[
\left( \sum_{n=1}^{k} (x_n + y_n)^p \right)^{1/p} \leq \left( \sum_{n=1}^{k} x_n^p \right)^{1/p} + \left( \sum_{n=1}^{k} y_n^p \right)^{1/p}.
\]
(Hint: Write \( \sum_{n=1}^{k} (x_n + y_n)^p = \sum_{n=1}^{k} x_n^p + \sum_{n=1}^{k} y_n^p - \sum_{n=1}^{k} (x_n + y_n)^{p-1} (x_n + y_n) \) and use Hölder Inequality for each of the two sums.)
(iv) Show that for each \( p \geq 1 \), the function \( d_p : \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R} \)

\[
d_p(x, y) = \left( \sum_{n=1}^{k} |x_n - y_n|^p \right)^{1/p}
\]

is a metric on \( \mathbb{R}^k \).

**Problem II** For each \( 1 \leq p \leq \infty \) show that any sequence in \( (\mathbb{R}^k, d_p) \) is bounded if and only if all its component sequences are bounded.

**Problem III** Let \( l_\infty \) be the set of all bounded, real valued sequences \( x = (x_1, x_2, \ldots) \) and \( d_\infty(x, y) = \sup\{|x_j - y_j| \mid j = 1, 2, \ldots\} \). Show that:

(a) \( d_\infty \) is a metric on \( l_\infty \);
(b) a sequence \( x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots) \), \( n \in \mathbb{N} \), in \( (l_\infty, d_\infty) \) converges to \( x = (x_1, x_2, \ldots) \) if and only if for any \( \epsilon > 0 \) there is \( N \in \mathbb{N} \) such that \( |x_j^{(n)} - x_j| < \epsilon \) for all \( n > N \) and \( j = 1, 2, \ldots \) (in other words the component sequences are uniformly convergent);
(c) \((l_\infty, d_\infty)\) is a complete metric space;
(d) the sequence \( x^{(n)} = (0, 0, \ldots, 0, 1, 0, \ldots) \), \( n \in \mathbb{N} \), where 1 appears exactly in the \( n^{th} \) component is bounded in \( (l_\infty, d_\infty) \) but has no convergent subsequence (Bolzano-Weierstrass Property does not hold).

**Problem IV** Let \( l_1^c \) be the set of all real valued sequences which are eventually zero:

\[
l_1^c = \{x = (x_1, x_2, \ldots) \mid \exists N \in \mathbb{N} \ x_n = 0 \ \forall n > N\}
\]
and \( d_1(x, y) = \sum_j |x_j - y_j| \). Show that:

(i) \( d_1 \) is a metric on \( l_1^c \);
(ii) if a sequence \( x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots) \), \( n \in \mathbb{N} \), in \( (l_1^c, d_1) \) converges to \( x = (x_1, x_2, \ldots) \) then, for each \( j \in \mathbb{N} \), the component sequence \( x_j^{(n)} \) converges to \( x_j \);
(iii) the sequence \( x^{(n)} = (1, 1/2, \ldots, 1/2^n, 0, 0, \ldots) \), \( n \in \mathbb{N} \), is Cauchy in \( (l_1^c, d_1) \) but it is not convergent.

**Problem V** Show that in any metric space the Bolzano-Weierstrass property implies completeness.