Math 444 Midterm II
March 29, 2018

1. Let $a > 0$ and $s_1 > 0$. Construct the sequence $(s_n : n \in \mathbb{N})$ via

\[ s_{n+1} = \frac{3}{4}s_n + \frac{1}{4}s_n^3, \quad n \geq 1. \]

(i) (5 points) Show that $s_n > 0$, for all $n \in \mathbb{N}$ and $s_n^4 \geq a$, for all $n \geq 2$. (Hint: for the second use the arithmetic/geometric inequality $\left(\frac{x_1+x_2+x_3+x_4}{4}\right)^4 \geq x_1x_2x_3x_4$ valid for any non-negative numbers.)

(ii) (5 points) Show that $(s_n : n \in \mathbb{N})$ is a monotone sequence.

(iii) (10 points) Is this sequence convergent? If yes can you find its limit? Carefully explain your answers.

2. Let $x_1, x_2$ be real numbers. Construct the sequence $(x_n : n \in \mathbb{N})$ via

\[ x_{n+1} = \frac{1}{3}x_n + \frac{2}{3}x_{n-1}, \quad n \geq 2. \]

(a) (5 points) Show that $(x_n : n \in \mathbb{N})$ is a Cauchy sequence.

(b) (5 points) Is this sequence convergent? Carefully explain your answer.

3. Consider the sequence $(x_n : n \in \mathbb{N}) = \left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots\right)$. A more suggestive representation is:

\[
\begin{align*}
\frac{1}{2} & \downarrow \\
\frac{1}{3} & \rightarrow \frac{2}{3} \\
\frac{1}{4} & \rightarrow \frac{2}{4} \rightarrow \frac{3}{4} \\
\ldots & \\
\frac{1}{n} & \rightarrow \frac{2}{n} \rightarrow \frac{3}{n} \ldots \frac{n-1}{n} \\
\ldots &
\end{align*}
\]

(i) (10 points) Find $\limsup_{n \to \infty} x_n$ and $\liminf_{n \to \infty} x_n$. 
(ii) (5 points) Show that \((x_n : n \in \mathbb{N})\) has a subsequence converging to zero and a subsequence converging to 1.

(iii) (10 points) Show that \((x_n : n \in \mathbb{N})\) has a subsequence converging to \(L\) if and only if \(0 \leq L \leq 1\).

4. Determine whether the following series are convergent. Explain your answers.

(a) (10 points) \(\sum (-1)^n (1 - \frac{1}{n})^{n+1}\)

(b) (5 points) \(\sum \frac{n+1}{n!}\)

(c) (5 points) \(\sum (-1)^{n+1} (\frac{1}{n} - \frac{1}{n+1})\)

(d) (10 points) Show that \(\sum_{n=1}^{\infty} (\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}) = 1\). (Hint: write down the first few partial sums.)

5. (i) (5 points) Let \(A \subseteq \mathbb{R}\), let \(f, g : A \mapsto \mathbb{R}\) and let \(c\) be a cluster point for \(A\). Assume:

\[ f(x) < g(x) \quad \text{for all } x \in A, \]

and that both \(\lim_{x \to c} f(x) = L_1\) and \(\lim_{x \to c} g(x) = L_2\) exist. Show that \(L_1 \leq L_2\).

(ii) (5 points) Find an example in part (i) for which \(L_1 = L_2\).

(iii) (5 points) Let \(f, g : \mathbb{R} \mapsto \mathbb{R}\) where:

\[ f(x) = \frac{x^3 + 3x + 1}{(x^2 + 1)^2}, \quad g(x) = x^5. \]

Show that the limit \(\lim_{x \to 1} g(f(x))\) exists and calculate it.

6. (20 extra credit points) Consider a totally ordered field \(\mathbb{F}\). Assume it has the Bolzano-Weierstrass Property i.e., any bounded sequence in \(\mathbb{F}\) has a convergent subsequence. Show that both the Archimedean Property and the Completeness Property also hold. That is: for any \(f \in \mathbb{F}\) there is \(n \in \mathbb{N}\) such that \(f < n\), (note that a totally ordered field contains natural and rational numbers) AND any non-empty subset of \(\mathbb{F}\) which has an upper bound has a least upper bound (supremum).
1. (i) $S_n > 0$ by induction:

$S_1 > 0$ by hypothesis.

if $S_n > 0$ then

\[ \frac{3}{4} S_n > 0. \quad \therefore S_{n+1} = \frac{3}{4} S_n + \frac{1}{4} \frac{a}{S_n^3} > 0. \]

\[ \frac{1}{4} \frac{a}{S_n^3} > 0. \]

By induction $S_n > 0 \ \forall \ n \geq 1$

For $n \geq 2$:

\[ \begin{align*}
S_n^4 &= \left( \frac{\left( S_n + S_{n-1} + \frac{a}{S_{n-1}^3} \right)^4}{4} \right) \\
&\geq S_{n-1} \cdot S_{n-1} \cdot S_{n-1} \cdot \frac{a}{S_{n-1}^3} = a.
\end{align*} \]

where we used $S_{n-1} > 0 \ \forall \ n \geq 2$ and the geometric inequality.

(ii) $S_n - S_{n+1} = \frac{1}{4} S_n - \frac{1}{4} \frac{a}{S_n^3} = \frac{1}{4} \frac{S_n^4 - a}{S_n^3} \geq 0$ since

$S_n > 0$ and $S_n^4 \geq 0$ by part (i) \( \Rightarrow \) $S_{n+1} \leq S_n \ \forall \ n \geq 1$

\( \Rightarrow \) the sequence is decreasing.
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(iii) $(S_n : n \in \mathbb{N})$ is decreasing.
\[ S_n > 0 \quad \forall n \in \mathbb{N} \quad \text{by (i) so it is odd how below} \]
\[ \Rightarrow (S_n : n \in \mathbb{N}) \quad \text{is convergent to } L \geq 0 \]
\[ \Rightarrow (S_{n+1} : n \in \mathbb{N}) \quad \text{is also convergent to } L \]
\[ S_{n+1} = \frac{3}{4} S_n + \frac{1}{4} a \]
\[ L = \frac{3}{4} L + \frac{1}{4} \frac{a}{L^3} \quad \Rightarrow \quad L^4 = a \quad \Rightarrow \quad L = a^{\frac{1}{4}} \]

2. (a) $x_{n+1} - x_n = \frac{2}{3} (x_{n-1} - x_n) \quad \forall n \geq 2.$
\[ \Rightarrow \quad |x_{n+1} - x_n| \leq \frac{2}{3} |x_n - x_{n-1}| \quad \forall n \geq 2. \]
\[ 0 < \frac{2}{3} < 1 \]

The sequence is contractive $\Rightarrow$ it is Cauchy.

(b) Yes, any Cauchy seq of real numbers is convergent!
3. (i) Let $y_k = \sup \{ x_n : n \geq k \}, \quad k \geq 1$

Then $y_k = 1 \quad \forall k \in \mathbb{N}$ because 1 is one upper bound and

$\forall \varepsilon > 0 \exists N \geq k : \frac{N-1}{N} > 1 - \varepsilon$.

Hence $\lim \sup_{u \to \infty} x_u = \lim_{k \to \infty} y_k = 1$.

Let $z_k = \inf \{ x_n : n \geq k \}, \quad k \geq 1$

Then $z_k = 0 \quad \forall k \in \mathbb{N}$ because 0 is a lower bound and

$\forall \varepsilon > 0 \exists N \geq k : \frac{1}{N} < \varepsilon$.

Hence $\lim \inf_{u \to \infty} x_u = \lim_{k \to \infty} z_k = 0$.

(ii) Since $0 = \lim \inf u \cdot \inf x_u$ (see part (i)) then there exists a subsequence of $(x_n)_{n \in \mathbb{N}}$ convergent to $0$.

1 = $\lim \sup u \cdot \lim_{u \to \infty} x_u$ implies $\exists (x_{n_k})_{k \in \mathbb{N}}$ suberssion:

$\lim_{k \to \infty} x_{n_k} = 1$
(iii) If \( \exists (x_{n_k} : k \in \mathbb{N}) \) subseq such that
\[ \lim_{k \to \infty} x_{n_k} = L \]
then \( \liminf_{n \to \infty} x_n \leq L \leq \limsup_{n \to \infty} x_n \)

(i) \( 0 \leq L \leq 1 \)

Reciprocal: Let \( 0 \leq L \leq 1 \). Since \( Q \) is dense in \( \mathbb{R} \) there is \( \left( \frac{P_k}{2^k} \right)' \) such that:
\[ \left( \frac{P_k}{2^k} : k \in \mathbb{N} \right) \]
such that:
\[ P_k, 2^k \in \mathbb{N}, \quad 0 < \frac{P_k}{2^k} < 1, \quad k \in \mathbb{N} \quad \text{and} \quad \lim_{k \to \infty} \frac{P_k}{2^k} = L \]

To construct the subsequence of \( (x_n : n \in \mathbb{N}) \) we use
\[ x_{n_1} = \frac{P_1}{2^1} \]
\[ x_{n_2} = \frac{P_2}{2^2} \quad \text{if} \quad 2^2 > 2 \quad \text{otherwise} \]
\[ x_{n_2} = \frac{P_2 \cdot 2^1}{2^2 \cdot 2} \quad \text{if} \quad 2^2 > 2 \quad \text{otherwise} \]
\[ x_{n_k} = \frac{P_k \cdot 2^1 \cdots 2^k}{2^k \cdot 2^{k-1} \cdots 2} \quad \text{if} \quad 2^{k+1} > 2 \quad \text{otherwise} \]
This way the denominator of $x_{nk}$ < denominator of $x_{k+1}$

$\Rightarrow n_k < n_{k+1} \Rightarrow x_{nk}$ is a subsequence of $x_k$ and

$$\lim_{k \to \infty} x_{nk} = \lim_{k \to \infty} \frac{b_k}{2^k} = \frac{1}{2}.$$ 

4. (a) \((4 - \frac{1}{n})^{u+1} = \frac{\Gamma(u+1)}{\Gamma(u)} = \frac{(u-1)^{u+1}}{u}

= \frac{1}{(1 + \frac{1}{u-1})^{u-1}} \cdot \left(\frac{u-1}{u}\right)^{u-1}

\Rightarrow \lim_{u \to \infty} (1 - \frac{1}{u})^{u+1} = \frac{1}{e}, \quad 1 > 0.

Since \(\lim_{u \to \infty} |(-1)^u (1 - \frac{1}{u})^{u+1}| = \frac{1}{e} \neq 0\)

the series is divergent.

(b) \(\lim_{n \to \infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n \to \infty} \frac{n+2}{(n+1)!} = \lim_{n \to \infty} \frac{n+2}{n!} = 0 < 1\)

the series is convergent by the ratio test!
(c) \[ |(-1)^{n+1} \left( \frac{1}{n} - \frac{1}{n+1} \right)| = \frac{1}{n(n+1)} \leq \frac{1}{n^2} \quad \forall n \geq 1. \]

Since \( \sum \frac{1}{n^2} \) is convergent, by the comparison criterion, \( \sum (-1)^{n+1} \left( \frac{1}{n} - \frac{1}{n+1} \right) \) is absolutely convergent hence convergent.

(d) \[ \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right) + \ldots + \left( \frac{1}{\sqrt{m}} - \frac{1}{\sqrt{m+1}} \right) \]

\[ = 1 - \frac{1}{\sqrt{m+1}} \]

\[ \therefore \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) = \lim_{m \to \infty} \left( \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) = 1 \]

(E) (i) Let \( \left( x_n \right) \) (\( x_n \in \mathbb{A} : n \in \mathbb{N} \)) such that \( \lim_{n \to \infty} x_n = c \).

Consequently, \( \lim_{n \to \infty} \phi(x_n) = L_1 \) and \( \lim_{n \to \infty} \phi(x_n) = L_2 \)
But from \( f(x_n) < g(x_n) \) we deduce from properties of sequence:

\[
L_1 = \lim_{n \to \infty} f(x_n) \leq \lim_{n \to \infty} g(x_n) = L_2.
\]

(ii) Let \( f, g : (0, 1) \to \mathbb{R} \).

\[
f(x) = 1 - x < 1 = g(x) \quad \forall x \in (0, 1)
\]

and

\[
\lim_{x \to 0} f(x) = 1 = \lim_{x \to 0} g(x).
\]

(iii) Both \( f, g \) are continuous on \( \mathbb{R} \). Hence,

\[
g \circ f \text{ is continuous on } \mathbb{R} \text{ in particular at } 1.
\]

\[
= \lim_{x \to 1} g(f(x)) = g(f(1)) = g\left(\frac{5}{4}\right) = \left(\frac{5}{4}\right)^5
\]

6. If \( F \) has the Archimedean Property:

Assume \( \exists f \in F : f \geq u \quad \forall u \in N \) then

the sequence \( \{ u_n \} \) is bounded \((0\) is a lower bound and \( \infty \) an upper bound). By B-W property

\[
\exists \{ u_k \in \mathbb{N} \} \text{ subsequence convergent to } 0
\]
\[ \exists K \in \mathbb{N} : \quad f_0 - \frac{1}{2} < x_{n_k} < f_0 + \frac{1}{2} \quad \forall k \geq K \]

\[ \text{But Hence} \]
\[ f_0 - \frac{1}{2} + 1 < x_{n_K} < f_0 + \frac{1}{2} \]
\[ \Rightarrow x_{n_{K+1}} > f_0 + \frac{1}{2} \quad \text{contradiction.} \]

If \( \text{no the completeness property, let} \ A \subseteq \mathbb{F} \text{ nonempty and} \ b \in A \text{ such that} \]
\[ a \leq b \quad \forall a \in \mathbb{A}. \]

Fix \( a_1 \in \mathbb{A}. \) Construct \((a_n \in A : n \in \mathbb{N})\), \((b_n \in F : n \in \mathbb{N})\)
in the following way:

\[ a_1 = a_1, \quad b_1 = b. \]

If \( \exists a \in A : a > \frac{a_1 + b_1}{2} \) then \( a_2 = a_1, \ b_2 = b_1. \)

Otherwise \( a_2 = a_1, \ b_2 = \frac{a_1 + b_1}{2}. \)

If \( \exists a \in \mathbb{A} : a > \frac{a_n + b_n}{2} \) then \( a_{n+1} = a, \ b_{n+1} = b_n. \)

Otherwise \( a_{n+1} = a_n \) and \( b_{n+1} = \frac{a_n + b_n}{2}. \)
Consequently, \( \{a_n : n \in \mathbb{N}\} \) is included in \( A \) for \( (b_1 : n \in \mathbb{N}) \) is a sequence of upper bounds.

\[
0 \leq b_{n+k} - a_{n+k} \leq \frac{b_n - a_n}{2} \\
\forall n \in \mathbb{N} \Rightarrow 0 \leq b_{n+k} - a_{n+k} \leq \frac{b_n - a_n}{2n}
\]

Also, \( a \leq b_n \leq b \ \forall n \in \mathbb{N} \Rightarrow 3 \text{ (} b_{n_k} \text{ : } n \in \mathbb{N} \text{)} \)

Subsequence of \( (b_n : n \in \mathbb{N}) \) and \( \text{LEF} \):

\[
\lim_{k \to \infty} b_{n_k} = L.
\]

We show \( L = \sup A \):

\[
\text{If } \exists a \in \mathbb{A}: a > L \Rightarrow 3 \text{ } n \in \mathbb{N}: \begin{array}{c}
\overline{a} \\
\mid b_{n_k} - L \mid < a - L \end{array} \forall k \geq K
\]

\[
\Rightarrow b_{n_k} < a \Rightarrow b_{n_k} \text{ is not an upper bound for } \mathbb{A} \text{ } \text{Contradiction}
\]

\[
\text{If } \exists L_1 \in \mathbb{F}: L_1 < L \text{ and } L_1 \text{ an upper bound for } \mathbb{A}
\]

\[
\Rightarrow 3 \text{ } K \in \mathbb{N}: \overline{b_{n_k}} > \frac{L + L_1}{2} \forall k \geq K
\]

and from \( a_{n_k} \leq L \ \forall k \geq K \) we get:

\[
\frac{b - a}{2^{n_k-1}} \geq b_{n_k} - a_{n_k} \geq \frac{L - L_1}{2} \forall k \geq K
\]
6. (cont) Hence \( \forall k \geq K \)

\[
\frac{2(b-a)}{L-L_1} > 2^{\frac{u_k-1}{2}} \geq 1 + u_k - 1 = u_k \geq k
\]

\( = \) \( N \) is bounded by \( \max \{ 1, 2, \ldots, K, \frac{2(b-a)}{L-L_1} \} \)

Contradiction with Archimedean Property.

Hence \( L = \sup A \).