In the following problems you must show all your work for full credit!

1. (25 points) Find the solution of the problem:

\[ 3u_x + 6u_y = (y - 2x)u, \quad u(0, y) = y^2 \]

2. Consider the equation:

\[ -4u_{xx} + 8u_{xy} - 4u_{yy} + u_x + u_y = 0 \]

(i) (5 points) What type of equation is it?
(ii) (10 points) Reduce it to canonical form. (Hint: use an orthogonal system of new coordinates).
(iii) (10 points) Use the canonical form to find the solution that satisfies the condition: 
\[ u(x, -x) = e^{-4x^2}. \]

3. Consider the wave equation on the segment \( 0 \leq x \leq 2 \):

\[
\begin{align*}
    u_{tt} - 4u_{xx} &= 0, \\
    u(t, 0) &= \sin t, \quad u(t, 2) = 0, \\
    u(0, x) &= 0, \quad u_t(0, x) = 0. 
\end{align*}
\]

(a) (10 points) Use the energy method to show that the problem has a unique solution.
(b) (15 points) Find \( u(t = 1, x = 1) \).

4. (25 points) Let \( u \) satisfy:

\[ u_t - u_{xx} + u = 0, \quad t > 0, \quad 0 < x < 1; \]

and assume

\[ u(0, x) \leq 0, \text{ for } 0 \leq x \leq 1; \quad u(t, 0) \leq 0, \text{ for } t > 0; \quad u(t, 1) \leq 0, \text{ for } t > 0. \]

Show that \( u(t, x) \leq 0 \) for all \( t > 0 \) and \( 0 \leq x \leq 1 \).
1. \[ \frac{dy}{dx} = 2 \Rightarrow y - 2x = C \Rightarrow \frac{\partial y}{\partial x} = -\frac{1}{2} \Rightarrow 2y + x = C \Rightarrow \frac{\partial y}{\partial x} = 1 \Rightarrow h = x + 2y. \]

\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} (-2) + \frac{\partial u}{\partial h} \cdot 1 \]

\[ \frac{\partial u}{\partial y} = \frac{\partial u}{\partial h} \cdot 1 + \frac{\partial u}{\partial h} \cdot 2 \]

Plug in the eq

\[ 15 \frac{\partial u}{\partial h} = \zeta u \]

\[ \Rightarrow u(x_0, y) = C(x) e^\frac{\zeta}{15} h. \]

\[ \Rightarrow u(x, y) = C(y) e^\frac{2}{15} y^2. \]

\[ y^2 u(x_0, y) = C(y) e^\frac{2}{15} y^2 \Rightarrow C(y) = y^2 e^{-\frac{2}{15} y^2} \]

\[ \Rightarrow u(x, y) = (y - 2x)^2 e^{-\frac{2}{15} (y - 2x)^2} \]

\[ = (y - 2x)^2 e^{\frac{x(y - 2x)}{3}} \]

-2-
2. (i) \[ \Delta = 4^2 - (-4)(-4) = 0 \Rightarrow \text{parabola.} \]

(ii) \[-4 \frac{d^2}{dx^2} + 8 \frac{d}{dx} \frac{y}{z^2} - 4 \frac{d^2}{dz^2} = 0.\]

\[ \Rightarrow \left( \frac{d^2}{dx^2} \right)^2 - 2 \left( \frac{d}{dx} \frac{y}{z^2} \right) + 1 = 0 \]

\[ \Rightarrow \frac{d}{dx} \frac{y}{z^2} = 1 \]

\[ \Rightarrow \frac{dy}{dx} = -\frac{d}{dz} = -1 \Rightarrow x' = x + y. \]

For orthosolvility use \[ y' = x - y. \]

\[ \frac{\partial U}{\partial x} = \frac{\partial U}{\partial x'} + \frac{\partial U}{\partial y'} \]

\[ \frac{\partial U}{\partial y} = \frac{\partial U}{\partial x'} - \frac{\partial U}{\partial y'} \]

\[ \frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial x'^2} + 2 \frac{\partial^2 U}{\partial x' \partial y'} + \frac{\partial^2 U}{\partial y'^2} \]

\[ \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial x'^2} - \frac{\partial^2 U}{\partial y'^2} \]

\[ \frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 U}{\partial x'^2} - 2 \frac{\partial^2 U}{\partial x' \partial y'} + \frac{\partial^2 U}{\partial y'^2} \]
Plug in the equation:

\[-16 \frac{\partial^2 u}{\partial y'^2} + 2 \frac{\partial u}{\partial x'} = 0\]

or \( \frac{\partial u}{\partial x'} = 8 \frac{\partial^2 u}{\partial y'^2} = 0 \) (treat corn in variable \( t = x', \ x = y' \)).

(iii). Initial condition is along the curve \( y = -x \)
\( y + x = 0 \Rightarrow x' = 0 \), so.

\[ \begin{cases} 
\frac{\partial u}{\partial x'} - 8 \frac{\partial^2 u}{\partial y'^2} = 0 & x' > 0, \ y' \in \mathbb{R} \ \\
U(0, y') = e^{-y'^2} = \varphi(y') \end{cases} \]

\[ U(x', y') = \int_{-\infty}^{\infty} S(x', y' - \xi) \varphi(\xi) \, d\xi \]

where \( S(x', y') = \frac{1}{\sqrt{4.8 \pi t}} e^{-\frac{y'^2}{4.8t}} \)

In this case one can explicitly calculate the solution.
\[ U(x', y') = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi x'}} \ e^{-\frac{(y' - 2)^2}{32x'}} \ e^{-\frac{z^2}{2}} \, dz. \]

\[ = \frac{1}{4 \sqrt{2\pi x'}} \int_{-\infty}^{\infty} \ e^{-\frac{2(1 + 32x') y'^2}{1 + 32x'} \ e^{-\frac{(y' - (1 + 32x')^2 / 32x')}}} \, dz. \]

\[ = \frac{1}{\sqrt{1 + 32x'}} \ \text{if} \ x' > 0, \ y' \in \mathbb{R}. \]

\[ \Rightarrow U(x, y) = \frac{1}{\sqrt{1 + 3(x+y)}} \ e^{-\frac{(x+y)^2}{1 + 32(x+y)}} \ \text{if} \ y > -x. \]

3. (i) Let \( U, V \) be two solutions. Then

\[ W = U - V \] satisfies

\[ \begin{cases} W_{tt} - 4 W_{xx} = 0 \\ W(t, 0) = 0 = W(t, 2) \\ W(0, x) = 0, \ W_t(0, x) = 0. \end{cases} \]

Multiply the 2nd by \( W_t \) and integrate from 0 to 2 in \( x \):

\[ \int_0^2 W_{tt} W_t \, dx - 4 \int_0^2 W_{xx} W_t \, dx = 0. \] Integrate by parts the second integral

\[ = \left. \frac{1}{2} W_t^2 \right|_0^2 - 4 \int_0^2 W_t \, W_t \, dx = 0. \]
\[ w_t(t, 0) = \frac{\partial}{\partial t} w(t, 0) = \frac{\partial}{\partial t} 0 = 0 \]

\[ w_{t}(t, 2) = \frac{\partial}{\partial t} w(t, 2) = \frac{\partial}{\partial t} 0 = 0 \]

\[ \Rightarrow w_x w_t \bigg|_{0}^{2} = 0 \]

\[ \Rightarrow \frac{1}{2} \frac{\partial}{\partial t} \left[ \int_{0}^{2} w_t^2 \, dx + 4 \cdot \frac{1}{2} \frac{\partial}{\partial t} \int_{0}^{2} w_x^2 \, dx \right] = 0 \]

\[ \Rightarrow \frac{\partial}{\partial t} \left[ \frac{1}{2} \int_{0}^{2} (w_t^2 + 4w_x^2) \, dx \right] = 0 \]

\[ \text{Energy} = E(t) \]

\[ \Rightarrow B(t) = \text{constant} ; \quad E(0) = \frac{1}{2} \int_{0}^{2} w_t^2(0, x) + 4 \left( \frac{\partial}{\partial x} w(0, x) \right)^2 dx \]

\[ = \frac{1}{2} \int_{0}^{2} 0^2 + 4 \cdot 0^2 \, dx = 0. \]

\[ \Rightarrow E \equiv 0 \Rightarrow \int_{0}^{2} w_t^2 + 4w_x^2 \, dx = 0 \Rightarrow \begin{cases} w_t \equiv 0 \\ w_x \equiv 0 \end{cases} \]

\[ \Rightarrow \text{W = constant since } w(0, x) \equiv 0 \Rightarrow W \equiv 0 \]

\[ \Rightarrow U \equiv V = ) \text{ solution is unique} \]

\((\ddagger) \quad \begin{cases} U(t, x) = V(t, x) + \frac{1}{2} \text{ Solve } (x-2) \end{cases} \)

\[ \Rightarrow \begin{cases} U_{tt} - 4 U_{xx} = -\frac{1}{2} \text{ Solve } (x-2) \\ U(t, 0) = 0 = U(t, 2) \\ U(0, x) = 0, \quad U_t(0, x) = \frac{1}{2} \text{ Solve } (x-2). \end{cases} \]
\[ U(t,1) = \frac{1}{4} \int_{-1}^{3} \frac{\psi(x)}{2} (x^2) \, dx + \frac{1}{4} \int_{1-2(1-c)}^{1+2(1-c)} \int_{0}^{\frac{\pi}{2}} \sin(t) \, dx \, d\alpha \]

where \[
\psi(x) = \begin{cases} 
\frac{1}{2} (x-2) & 0 < x < 2 \\
-\frac{1}{2} (-x-2) & -2 < x < 0 & \text{(odd extension)} \\
\frac{1}{2} (-x+4-2) & 2 < x < 6 & \text{(periodic)} \\
\frac{1}{2} (x-2) & 2 < x < 6 & \text{(periodic)} 
\end{cases} 
\]

\[
\hat{f}(t,x) = \begin{cases} 
-\frac{1}{2} \sin(t) (x-2) & 0 < x < 2 \\
+\frac{1}{2} \sin(t) (-x-2) & -2 < x < 0 & \text{odd extension} \\
+\frac{1}{2} \sin(t) (-x+4-2) & 2 < x < 6 & \text{periodic} 
\end{cases} 
\]

\[
\int_{-1}^{3} \psi(x) \, dx = 0 \quad \text{(integral over a period of an odd function)}.
\]

The second integral can be decomposed into regions:

\[
\frac{1}{4} \int_{0}^{1} \int_{1-2(1-c)}^{1+2(1-c)} \hat{f}(t,x) \, dx \, d\alpha = \frac{1}{4} \int_{0}^{1-2(1-c)} \int_{1-2(1-c)}^{1+2(1-c)} \sin(t) \, dx \, d\alpha + \frac{1}{2} \int_{0}^{1} \sin(t) (x-2) \, dx \, d\alpha 
\]
\[
+ \frac{1}{4} \int_0^{1/2} \left( -\frac{1}{2} (\sin C) (x+2) \, dx \, dC + \frac{1}{1+2(1-C)} \int_{1/2}^{1+2(1-C)} \right) \left( -\frac{1}{2} (\sin C) (x-2) \, dx \, dC 
\right)
\]

\[
+ \frac{1}{4} \int_0^{1/2} \frac{1}{2} (\sin C) (x+2) \, dx \, dC
\]

\[
= \frac{1}{16} \int_0^{1/2} (4C^2 + 4C - 3) \sin C \, dC + \frac{1}{16} \int_0^{1/2} (4C - 2 \sin C + 1) \sin C \, dC - \frac{1}{16} \int_0^{1/2} (4C - 2 \sin C) \sin C \, dC
\]

\[
= \frac{1}{2} \int_0^{1/2} \sin C \, dC - \frac{1}{2} \int_0^{1/2} (C - 1) \sin C \, dC
\]

\[
= \frac{1}{2} \left( \sin C - (\cos C) \right) \bigg|_0^{1/2} + \frac{1}{2} \left[ (C - 1) \cos C - \sin C \right] \bigg|_0^{1/2}
\]

\[
= -\frac{1}{2} \sin 1 + \sin \frac{1}{2}
\]

\[
\therefore \quad U(1,1) = \sin \frac{1}{2} - \frac{1}{2} \sin 1
\]
4. Assume there is \( t_0 > 0, x_0 \in [0,1] \) such that \( u(t_0, x_0) > 0 \).

Then \( \max u(t, x) > 0 \) on \( (t, x) \in [0, t_0] \times [0,1] \).

Let \( (t_1, x_1) \) be the point where \( u \) reaches the max.

\[
\begin{align*}
\frac{\partial u}{\partial t}(t_1, x_1) &> 0; \\
\frac{\partial u}{\partial x}(t_1, x_1) &= 0; \\
\frac{\partial^2 u}{\partial x^2}(t_1, x_1) &\leq 0.
\end{align*}
\]

Thus, \( u(t_1, x_1) > 0 \).

\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u &> 0 \text{ at } (t_1, x_1) \text{ which contradicts the equation satisfied by } u = c.
\end{align*}
\]

Thus, no point \( t \geq 0, x \in [0,1] \); \( u(t, x) > 0 \).

\[
\begin{align*}
\Rightarrow u(t, x) &\leq 0 \uparrow.
\end{align*}
\]

**Note**: One cannot apply directly the weak maximum principle unless we require that \( u \) satisfies \( u_t - u_{xx} = 0 \).