I. Properties of general groups.  1. Define groups and abelian groups.

2. Show that in a group \((G, \cdot)\):

   (a) the neutral element is unique;
   (b) the inverse of an element is unique;
   (c) \((a^{-1})^{-1} = a\), \((ab)^{-1} = b^{-1}a^{-1}\);
   (d) \(\forall a, b, c \in G\) the equations \(ax = b\), \(xa = b\) have a unique solution, in particular the following simplification rules hold:

   \[ab = ac\] implies \(b = c\) and \(ba = ca\) implies \(b = c\).

3. Which properties of a binary operations can be easily read from its table? Explain how these properties are deduced from the table.


2. State and prove the Subgroup Criterion Theorem (including the part regarding finite subgroups).

3. If \((G, \cdot)\) is a group and \(x \in G\), \(|x| = n\) show that \(\{x^m \mid m \in \mathbb{Z}\}\) is a subgroup of \(G\) of order \(n\).

4. Prove that the subgroup relation is transitive.

5. If \((G, \cdot)\) is a group and \(A \subseteq G\), define \(Z(G)\), \(C_G(A)\) and \(N_G(A)\). Show (directly) \(Z(G) \leq C_G(A) \leq N_G(A) \leq G\), and, if \(G\) is abelian, \(Z(G) = C_G(A) = N_G(A) = G\).

6. Show that \(Z(G) = \bigcap_{g \in G} C_G(\{g\})\).

7. Prove that arbitrary intersections of subgroups of a given group is also a subgroup.

8. Define the subgroup generated by a subset of a group.

9. If \((G, \cdot)\) and \(A \subseteq G\) show that:

   \[\langle A \rangle = \{a_1^{e_1}a_2^{e_2}\ldots a_n^{e_n} \mid n \in \mathbb{Z}, \; n \geq 0, \; a_i \in A, \; e_i = \pm 1 \text{ for all } i\}\]
   \[= \{a_1^{a_1}a_2^{a_2}\ldots a_n^{a_n} \mid n \in \mathbb{N}, \; a_i \in A, \; \alpha_i \in \mathbb{Z}, \; a_i \neq a_{i+1} \text{ for all } i\},\]

   in particular if \(A = \{x\}\) then \(\langle A \rangle = \{1, x, x^2, \ldots, x^{n-1}\}\) if \(|x| = n < \infty\) and \(\langle A \rangle = \{x^m \mid m \in \mathbb{Z}\}\) if \(|x| = \infty\).
10. If \((G, \cdot)\) and \(A = \{a_1, a_2, \ldots, a_k\} \subseteq G\) such that \(a_i a_j = a_j a_i, \ \forall i, j = 1, 2, \ldots, k\), show that \(\langle A \rangle\) is abelian and \(|\langle A \rangle| \leq d_1 d_2 \ldots d_k\) where \(|a_i| = d_i, \ i = 1, 2 \ldots k\).

I.2. Homomorphisms and Isomorphisms. 1. Define homomorphisms and isomorphisms between two groups.

2. If \(\phi : G \rightarrow H\) is a group homomorphism show that
   (a) \(\psi(1_G) = 1_H\),
   (b) \(\phi(g^{-1}) = \psi(g)^{-1}, \ \forall g \in G\),
   (c) \(\phi(g^n) = \phi(g)^n, \ \forall n \in \mathbb{Z}\),
   (d) \(\ker \phi \leq G\),
   (e) \(\phi(G) \leq H\).

3. If a presentation of the group \(G\) is given, prove that \(\phi : G \rightarrow H\) is an homomorphism if and only if the images of the generators of \(G\) satisfy in \(H\) the same relations the generators satisfy in \(G\).

4. Show that if two groups have the same presentation then they are isomorphic.

I.3. Group Actions 1. Define group action of \(G\) on \(A\).

2. Define the stabilizer \(G_a\) of \(a \in A\). Show that \(G_a \leq G\).

3. Define the kernel of an action. Show that it is a subgroup of \(G\).

4. Show that an action induces an homomorphism between the group and the symmetric group of the set. Moreover the kernel of this homomorphism is exactly the kernel of the action.

5. Viceversa, show that any homomorphism \(\sigma : G \rightarrow S_A\) induces an action of \(G\) on \(A\).

6. Show that an action is faithful if and only if its kernel is \(\{1\}\).

7. Use group actions to show \(Z(G) \leq C_G(A) \leq N_G(A) \leq G\).

8. Show that an action of \(G\) on \(A\) induces an equivalence relation on \(A\) for which the equivalence classes are the orbits.

9. Prove Lagrange’s Theorem: if \(H \leq G\) and \(|G| < \infty\) then \(|H|\) divides \(|G|\). Use the left action of \(H\) on \(G\) and its orbits.

II. Examples of groups and their properties. 1. Find a presentation of the dihedral group \(D_{2n}, \ n \geq 3\).
2. Show that two permutations commute if and only if their cycle decompositions have disjoint cycles.

3. Show that the order of a permutation is the l.c.m of the length of the cycles in its cycle decomposition.

4. By choosing an action of $D_{2n}$ on $\{1, 2, \ldots, n\}$, $n \geq 3$ show that $S_n$ has a subgroup isomorphic with $D_{2n}$. In particular prove that $D_6 \cong S_3$.

5. Construct an injective homomorphism from $D_{2n}$ to $GL_2(\mathbb{R})$. Deduce that the latter has a subgroup isomorphic with $D_{2n}$.

6. Find the centralizer and normalizer of each element in $S_3$, $D_8$, $Q_8$, then find the center of each of these groups.

II.1 Cyclic Groups. 1. Define cyclic groups.

2. Prove that all cyclic groups of order $n$ are isomorphic with $\mathbb{Z}/n\mathbb{Z}$.

3. Prove that all cyclic groups of infinite order are isomorphic with $\mathbb{Z}$.

4. Prove that any subgroup of a cyclic group is cyclic.

5. If $|\langle x \rangle| = \infty$ then all subgroups of $\langle x \rangle$ are of the form $\langle x^m \rangle$, $m \in \mathbb{N} \cup \{0\}$. Moreover distinct $m$’s give distinct subgroups.

6. If $|\langle x \rangle| = n < \infty$ then the order of any subgroup of $\langle x \rangle$ is a divisor of $n$. Moreover, for each divisor $d$ of $n$ there is exactly one subgroup of $\langle x \rangle$ of order $d$, namely $\langle x^{n/d} \rangle$.

II.2 Lattice of Subgroups Prove that $\mathbb{Z}/p^n\mathbb{Z}$, $p$ prime, $\mathbb{Z}/12\mathbb{Z}$, $S_3$, $D_8$, $Q_8$ have the lattices shown on pages 68 - 69 in the textbook.

Section 2.5 (Textbook pages 71 - 72,) Mandatory Problems: 11 - 15; Suggested problems: 1 - 10, 16 - 20.