Homework 1 comments:

1. $x(0) = 0$ makes no sense!
   $x = 0$ means $x(t) = 0$ for all $t$

2. Solution and comments for textbook problem 2.6 at page 17:

   $y(t) = -4.9 \ t^2 + v_0 \ t + y_0$  \(\tag{1}\)

   $v(t) = \frac{dy}{dt}$

   $v(t) = -9.8 \ t + v_0$  \(\tag{2}\)

   Eliminate $t$ from (1) and (2):

   $y = \left( \frac{v_0 - v}{9.8} \right)^2 \cdot (-4.9) + v_0 \ \frac{v_0 - v}{9.8} + y_0$

   $y - y_0 = -\left( \frac{v_0 - v}{9.8} \right)^2 + v_0 \ \frac{v_0 - v}{9.8}$

   $\frac{2 (9.8) \ (y - y_0)}{2 (9.8)} = -\left( v_0^2 + v^2 - 2 v_0 v \right) + 2 v_0^2 - 2 v_0 v$

   $\frac{2 (9.8) \ (y - y_0)}{2 (9.8)} = v_0^2 - v^2$

   $v^2 = v_0^2 - 2 (9.8) (y - y_0)$  \(\text{Cons of Energy}\)
In general for constant acceleration $a$:

$$
\begin{cases}
 y(t) = \frac{a}{2} t^2 + v_0 t + y_0 \\
v(t) = a t + v_0 \\
v^2 = v_0^2 + 2 a (y - y_0)
\end{cases}
$$

Maximum height: Find max of $y$ from (1)

For $y(x) = ax^2 + bx + c$ \( a < 0 \)

maximum is obtained at

$$
 x = -\frac{b}{2a} \quad \left( y'(x) = 0 \implies x = -\frac{b}{2a} \right)
$$

and $\max y = y\left(-\frac{b}{2a}\right) = c - \frac{b^2}{4a}$

In our problem $a = -9.8 / 2 \quad b = 100 \quad c = 20$

$$
\implies y = 20 - \frac{100^2}{2 \times (-9.8)} = 20 + \frac{100^2}{2 \times 9.8}
$$
Time to get back to:

\[ y(t_b) = 20 \] and from (2)

\[-9.8 \frac{b_b^2}{2} + 100 b_b + 200 = 20\]

\[ b_b (100 - 4.9 b_b) = 0 \]

\[ b_b = 0 \] (the storing point) root good

\[ b_b = \frac{100}{4.9} \]

Time to hit the ground \( t_{\text{max}} \)

\[ y(t_{\text{max}}) = 0 \]

\[-9.8 \frac{t_{\text{max}}^2}{2} + 100 t_{\text{max}} + 20 = 0\]

(from roots of second degree eq)

\[ t_{\text{max}} = \frac{-100 \pm \sqrt{100^2 + 40 \times 9.8}}{-9.8} \]

Choose the positive result. What is the meaning of the negative one?
4 Review Exact Soln's

(i) Separable Eqn's

(ii) First order linear Eq

(iii) Bernoulli Eqn

(iv) General Method of Substitution

Quiz 1: Find all soln's of the eqn

\[ x^2 y' + xy = 0 \]

(v) Homogeneous Eqn

See Lecture 5

(vi) Exact Eqn

\[ \frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)} \]
Find the solution in the implicit form:

\[ F(x, y) = C \]

How do we find \( F \)? Along the line of the stuff given:

\[ F(x, y(x)) = C \]

\[ \frac{d}{dx} F(x, y(x)) = 0 \]

Chain Rule:

\[ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0 \]

\( y \) is a soln: \( \Rightarrow \frac{dy}{dx} = - \frac{P(x, y)}{Q(x, y)} \)

Plug in above:

\[ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \left[ - \frac{P(x, y)}{Q(x, y)} \right] = 0 \]
One possibility is

\[
\begin{cases}
\frac{\partial F}{\partial x} = P(x, y) \\
\frac{\partial F}{\partial y} = Q(x, y)
\end{cases}
\]

From calculus, for smooth functions,

\[
\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}
\]

compatibility condition

If compatibility condition is satisfied, one uses calculus to solve \((x, y)\) and get the general solution in the form

\[
F(x, y) = C, \quad C \in \mathbb{R}
\]
If compatibility condition is not satisfied then (8) does not imply (9) but

\[ \frac{\partial F}{\partial x} = p(x, y) S(x, y) \]

\[ \frac{\partial F}{\partial y} = q(x, y) S(x, y) \]

Such that

\[ \frac{\partial}{\partial y} (pS) = \frac{\partial}{\partial x} (qS) \]

S is called an integrating factor and is usually very hard to find. There are special situations in which one can easily construct S, see book by:

KAMKE, E. : Differentialgleichungen I, 1977

POLYANIN, A. I. and ZAITSEV, V. F. Handbook of

exact sln's for ODEs, 2003

We will focus on the exact equation

or, in other words, S is not needed!
Example 12

\[ \frac{dy}{dx} = \frac{x^2 + y^2}{-2xy + 1} \]

To check for exactness write it in the form:

\[ -(x^2 + y^2) \, dx + (-2xy + 1) \, dy = 0 \]

\[ P(x, y) \quad Q(x, y) \]

\[ \frac{\partial}{\partial y} (-x^2 - y^2) = -2y \quad \text{exact eqn} \]

\[ \frac{\partial}{\partial x} (-2xy + 1) = -2y \]

\underline{Method I to find F:}

\[ \frac{\partial F}{\partial x} = P(x, y) \Rightarrow F(x, y) = \int P(x, y) \, dx + g(y) \]

\[ = -\frac{x^3}{3} - y^2 x + g(y) \]
\begin{align*}
\frac{\partial}{\partial y} f(x, y) &= q(x, y) \\
\Rightarrow -2y' x + g'(y) &= -3xy + 1 \\
\Rightarrow g'(y) &= 1 \quad \Rightarrow \quad g(y) = y \\
\Rightarrow f(x, y) &= -\frac{x^3}{3} - y^2x + y
\end{align*}

The general form is:
\[ f(x, y) = C \]
\[ -\frac{x^3}{3} - y^2x + y = C \]

Method II to find \( F \):

Fix a point in the plane \((x_0, y_0)\). Integrate along any path connecting \((x_0, y_0)\) with \((x, y)\).
\[ F(x, y) = \int_y^{x} (x^2 + y^2) \, dx + (-2x + 1) \, dy \]

or in general

\[ F(x, y) = \int_y^{x} P(x, y) \, dx + Q(x, y) \, dy \]

I'll choose \((x_0, y_0) = (0, 0)\) and the following simple path:

\[ F(x, y) = \int_0^x (t^2 + 0) \, dt + \int_0^y (2xt + 1) \, dt \]

\[ = -\frac{x^3}{3} - xy^2 + y \]
The solution of the problem is
\[-\frac{x^3}{3} - xy^2 + y = C\]

Check \( y = \ldots \), \( \frac{dy}{dx} = \ldots \) plug in the eqn. This step is not required as long as you check the Compatibility Condition and integrate correctly you should get the correct solution

(viii) Second order reducible again

\( F(x, y', y'') = 0 \)

\( u = y' \Rightarrow y'' = u' = \Rightarrow F(x, u(x), u'(x)) = 0 \)

first order eqn that might be solved with techniques (v)-(vi)

**Example 13:** \( y'' = \frac{y'}{x} \)

\( u = y' \), \( y'' = u' = \Rightarrow u' = \frac{u}{x} \) Separable