Summary

5. Direction (Slope) Field
6. Existence and Uniqueness

Theorem
4. Exact solution (Continuation)

5. Direction Field

We studied it and its application in understanding the behavior (properties) of solutions of first order ODEs in Lab 1 = Lecture 3.

On the next picture we will review its basic principles:

- How were the tiny segments obtained?
- How do you draw a solution on a slope field plot?
- What special features of the solutions and the eg can be inferred from the next plot?
\[ y' = y(3 - y) \]
6. Existence and Uniqueness Theorem

Consider the initial value problem

\[
\begin{align*}
   y'(x) &= f(x, y) \\
   y(a) &= b
\end{align*}
\]

Assume there exists a rectangle

\[ R = \{ (x, y) \mid x_1 < x < x_2, \ y_1 < y < y_2 \} \]

such that

1. \( (a, b) \in R \)
2. \( f \) is continuous on \( R \)
3. \( \frac{\partial f}{\partial y} \) is continuous on \( R \)

Then the initial value problem has a unique solution in the rectangle \( R \).

Proof is not required (see the Appendix in the textbook)

Recall

**Example 1** \( y' = ky \)

**Claim:** all solutions are given by \( y(t) = Ce^{kt} \), \( C \in \mathbb{R} \)

**Example 5** \( xy' = 2y \)

**Claim:** all solutions are given by \( y = Cx^2, \ C \in \mathbb{R} \)
Applications of the Theorem

Example 1 \[ f(x, P) = kP \]
\[ \frac{\partial f}{\partial P} = k \]

Both functions are continuous on \( \mathbb{R}^2 \)
\( \Rightarrow \) through any initial condition \( P(t_0) = P_0 \)

presents a unique slw. But this slw can be obtained from
\[ P(t) = C_2 e^{kt} \]

by choosing \( C_2 = \frac{P_0}{e^{-kt_0}} \). So there cannot be many other slw's

Example 5 \[ y' = 2 \frac{y}{x} \]
\[ f(x, y) = 2 \frac{y}{x} \]
\[ \frac{\partial f}{\partial y} = \frac{2}{x} \]
Both are continuous on $x 
eq 0$
Assume there exist a slu $\tilde{y} : I \rightarrow IR$
not given by

$$y(x) = C_2 x^2 \quad C_2 \in IR$$

Pick $x_0 \in I \setminus \{0\}$ and let $y_0 = \tilde{y}(x_0)$. The theorem applies on the rectangle

$$R = (-\infty, 0) \times IR \quad \text{if } x_0 < 0$$

$$R = (0, \infty) \times IR \quad \text{if } x_0 > 0$$

$\Rightarrow$ there is only one slu such that

$$y(x_0) = y_0$$

and since $\tilde{y}$ is such a slu we have

$$y = \tilde{y}$$

but $\tilde{y}(x) = C_2 x^2$ with $C_2 = y_0 / x_0^2$

is also a slu satisfying $y(x_0) = y_0$. So $\tilde{y}$ is given by

$$\tilde{y}(x) = C_2 x^2 \quad C_2 = y_0 / x_0^2$$

**Contradiction !!!**
Important Note: The theorem does not apply for the initial data:

\[ y(0) = y_0 \]

and indeed its conclusion is false:
- for \( y_0 \neq 0 \) there are no solutions
- for \( y_0 = 0 \) there are infinitely many such:

\[ y(x) = cx^2 \text{ for any } c \in \mathbb{R} \]

**Example 6** \[ y' = -2\sqrt{y} \quad y > 0 \]

Claim: all saturated solutions are of the form:

\[ y = 0 \]

or, for \( c \in \mathbb{R} \)

\[ y = \begin{cases} (c - x)^2 & x < c_1 \\ 0 & x \geq c_1 \end{cases} \]
\[ f(x, y) = -2\sqrt{y} \quad \text{continuous on } y > 0 \]

\[ \frac{\partial f}{\partial y} (x, y) = -\frac{1}{\sqrt{y}} \quad \text{continuous on } y > 0 \]

Theorem applies on \((x, y) \in \mathbb{R} \times (0, \infty)\)

Suppose \(\tilde{y} : I \to \mathbb{R}\) is a solution

1. If there is no \(x_0 \in \mathbb{R}\) such that \(\tilde{y}(x_0) \neq 0\) then \(\tilde{y} \equiv 0\)

2. If there is \(x_0 \in \mathbb{R}\) such that \(\tilde{y}(x_0) \neq 0\) \(\Rightarrow \tilde{y}'(x_0) > 0\)

\(\Rightarrow (x_0, \tilde{y}(x_0)) \in \mathbb{R} \times (0, \infty)\) and by the theorem there is a unique solution passing through \((x_0, \tilde{y}(x_0))\). Thus

\[ y(x) = (C - x)^2 \quad x < C \]

with

\[ C = x_0 + \sqrt{\tilde{y}(x_0)} \]

such a solution.

So \(\tilde{y}\) and \((C-x)^2\) coincide on \(x < C\).
By continuity \( \tilde{y}(c) = 0 \)

If \( \tilde{y}(x_i) > 0 \) for some \( x_i > c \)

Then \( (x_i, \tilde{y}(x_i)) \in \mathbb{R} \times (0, \infty) \) and as before

\[
\tilde{y} = (c_1 - x)^2 \quad x < c_1,
\]

\[
c_1 = x_i + \sqrt{\tilde{y}(x_i)} > c, \text{ because } x_i > c.
\]

Note for \( x < c \) we have 2 values for \( \tilde{y} \):

\[
\tilde{y} = (c - x)^2
\]

and

\[
\tilde{y} = (c_1 - x)^2 \quad c_1 > c.
\]

This is a contradiction, so \( \tilde{y}(x_i) = 0 \) for \( x \geq c \)!
4. (Continuation) Exact Solution

(ii) Linear first order ODE

\[ y'(x) + P(x) y(x) = Q(x) \]

**Example 7**

\[ y'(x) + x y(x) = x \]

**Integrating Factor**

\[ s(x) = e^{\int P(x) \, dx} \]

\[ s(x) = e^{\frac{x^2}{2}} \text{ for the example 7} \]

Multiply the eq by the integrating factor \((\neq 0)\)

\[ y'(x) \, s(x) + P(x) \, s(x) \, y(x) = Q(x) / s(x) \]

Observe that left hand side is:

\( (s(x) \, y(x))' \)
So,

\[
\begin{align*}
(y) & \quad (\mathcal{P}(x) \mathcal{Q}(x))' = \mathcal{P}(x) \mathcal{Q}(x) \\
\mathcal{P}(x) \mathcal{Q}(x) &= \int \mathcal{P}(x) \mathcal{Q}(x) \, dx + C \\
\Rightarrow y(x) &= \frac{1}{\mathcal{P}(x)} \left[ \int \mathcal{P}(x) \mathcal{Q}(x) \, dx + C \right]
\end{align*}
\]

For example,

\[
\begin{align*}
\mathcal{P}(x) &= e^{x^2/2} \\
y(x) &= e^{-x^2/2} \left[ \int e^{x^2/2} \, dx + C \right] \\
&= e^{-x^2/2} \left[ \frac{e^{x^2/2}}{x} + C \right] \\
&= 1 + Ce^{-x^2/2}
\end{align*}
\]

If there is an initial condition, say:

\[ y(0) = 2 \]

Then \[2 = y(0) = 1 + Ce^{-0^2} = 1 + C \Rightarrow \]
$C = 1$ and the clue of the eq. initial condition is

$$y(x) = 1 + e^{-x^2/2}$$

Another way to solve the linear first order ODE + initial condition

$$y(x_0) = y_0$$

is to use the formula

$$y(x) = \frac{1}{s(x)} \left[ y_0 + \int_{x_0}^{x} s(t) Q(t) \, dt \right]$$

which can be obtained from $(x)$ by using the definite integral $\int_{x_0}^{x} dt$.

Example 8

$$\frac{d^2y}{dx^2} + xy = x \sin x \quad x > 0$$

$$y(1) = 1$$

Rewrite $\frac{dy}{dx} + \frac{1}{x} y = \frac{\sin x}{x}$
\[ \varphi(x) = e^{\int \frac{1}{x} \, dx} = e^{\ln x} = x \]

\[ y(x) = \frac{1}{x} \left( 1 + \int_{1}^{x} \frac{\sin(t)}{t} \, dt \right) \]

\[ = \frac{1}{x} \left( 1 + \int_{1}^{x} \sin(t) \, dt \right) \]

\[ = \frac{1}{x} \left( 1 - \cos x \Big|_{1}^{x} \right) \]

\[ = \frac{1}{x} \left( 1 - \cos x + \cos 1 \right) \]