Summary:

1. Review of Def of F. S.
2. Convergence of Fourier Series
3. Properties of F. S.

1. Review of Definition of Fourier Series

Consider the real function \( f(t) \), \( t \in \mathbb{R} \), periodic with period \( P = 2L \). Then the following series

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n \pi}{L} t \right) + b_n \sin \left( \frac{n \pi}{L} t \right) \right]
\]

where

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \left( \frac{n \pi}{L} t \right) \, dt
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \left( \frac{n \pi}{L} t \right) \, dt
\]

is called the Fourier series associated with \( f(t) \).
Note: The formulas for coefficients $a_n$, $b_n$ are such that, for any $N > 0$, it minimizes

$$\int_{-L}^{L} \left( f(t) - \frac{a_0}{2} \sum_{n=1}^{N} \left[ a_n \cos \left( \frac{n\pi}{L} t \right) + b_n \sin \left( \frac{n\pi}{L} t \right) \right] \right)^2 dt$$

among all possible choices of $a_0, a_1, \ldots, a_N$ and $b_1, b_2, \ldots, b_N$.

See the table: simulation for the square wave function with period $2\pi$ when $N$ is 3, 25, 50, ...

The simulation also shows that the partial sums converge to the value of $f(t)$ at points where $f$ is continuous but not necessarily at discontinuity points.

The "mystery" is solved by the following convergence theorem:
2. Convergence of Fourier Series

Theorem. If \( f(t) \), \( t \in \mathbb{R} \) is piecewise \( C^1 \) and periodic, then its Fourier Series at any point \( t \in \mathbb{R} \) converges to

(i) \( f(t) \) if \( f \) is continuous at \( t \)

(ii) \( \frac{f(t^-) + f(t^+)}{2} \) if \( f \) is discontinuous at \( t \)

Definition. \( f(t) \), \( t \in \mathbb{R} \) is piecewise \( C^1 \) if on any bounded interval \([a, b]\) it has finitely many discontinuity points \( a < x_1 < \ldots < x_n \) and on each subinterval \([x_1, x_2], [x_2, x_3], \ldots, [x_n, b]\) the function is \( C^1 \) (it has continuous derivative and both the function and its derivative have finite limits at the endpoints, \( a, x_1, x_2, \ldots, x_n, b \)).

Examples: \( \sin t \) and \( \frac{1}{t} \) are piecewise \( C^1 \) when \( t \) are not! Why?

Square wave, triangular wave, sawtooth wave are!
Application: We know from Sec. 4.4 we have:

\[
\sum_{n=0}^{\infty} \left[ \frac{4}{\pi (2n+1)} \sin(2n+1)t \right] = f(t) = \begin{cases} 
-1 & -\pi < t < 0 \\
0 & t = 0, \pi \\
1 & 0 < t < \pi 
\end{cases}
\]

Then \( \mu = \frac{\pi}{2} \)

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1
\]

or

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}
\]

\[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots = \frac{\pi}{4}\]
3. Properties of Fourier Series

Consider the Fourier Series of a $P = 2L$ periodic function:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right]$$

Certain properties of $f$ are reflected in the coefficients $a_n, b_n$:

(i) Smoothness of $f$ is reflected in the decay of $a_n$ and $b_n$. See Project 5 and Lecture 17.

(ii) Symmetries of $f$:

- $f$ is odd then $a_n = 0$ for all $n$
- $f$ is even then $b_n = 0$ for all $n$

Examples: square and triangular wave = odd
$1/x$: extended $2\pi$ periodically = even
(iii) The F.S. of the derivative of $f$

If $f$ is continuous, piecewise $C^1$, $T=2\pi$ periodic and:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{2\pi n t}{T} \right) + b_n \sin \left( \frac{2\pi n t}{T} \right) \right]$$

then

$$f'(t) \sim \sum_{n=1}^{\infty} \left[ -a_n \frac{2\pi n}{T} \sin \left( \frac{2\pi n t}{T} \right) + b_n \frac{2\pi n}{T} \cos \left( \frac{2\pi n t}{T} \right) \right]$$

In other words, the F.S for $f'(t)$ can be obtained by differentiating term by term the F.S. for $f(t)$.

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Examples: the triangular wave

Counterexample: the square wave! It is not continuous so (iii) should not be used:

$$f(t) = \begin{cases} 
-1 & -\pi < t < 0 \\
0 & t = 0, \pi \\
1 & 0 < t < \pi
\end{cases} \quad f'(t) \equiv 0$$

$$f(t) = \sum_{n=1}^{\infty} \frac{4}{(2n+1)^2 \pi^2} \sin((2n+1)t)$$

Term by term diff:

$$f'(t) \sim \sum_{n=1}^{\infty} \frac{4}{(2n+1)^2 \pi^2} \cos((2n+1)t)$$

But all coeff of $f''(t)$ should be zero. So (iii) does not apply!
Proof of (iii): For simplicity assume $f$ is $2\pi$ periodic and it only has one point of discontinuity at $t_0 \in [-\pi, \pi]$. If the F.S. for the derivative is

$$f'(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos(nt) + b_n \sin(nt) \right)$$

Then by definition:

$$L_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(t) \cos(nt) \, dt =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f'(t) \cos(nt) \, dt + \frac{1}{\pi} \int_{-\pi}^{\pi} f'(t) \sin(nt) \, dt$$

by parts

$$= \frac{1}{\pi} \left[ \int_{-\pi}^{t_0} f(t_-) \cos(nt_0) \, dt - \int_{t_0}^{\pi} f(t_+) \cos(nt_0) \, dt \right]$$

$$+ \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \, dt$$

$$= m_n$$

Similarly, $b_n = -n a_n$ which is what term by term diff claims.

The argument can be easily extended for finitely many discontinuity points of the derivative and general periods.
(16) The F.S. of the antiderivative of \( f \).

If \( f \) is piecewise continuous, \( P = 2L \) periodic, and

\[
f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{2\pi nt}{L}\right) + b_n \sin\left(\frac{2\pi nt}{L}\right) \right]
\]

then

\[
F(t) \sim \frac{a_0 t}{2} + \sum_{n=1}^{\infty} \left[ \frac{a_n}{\pi n} \sin\left(\frac{2\pi nt}{L}\right) - \frac{b_n}{\pi n} \cos\left(\frac{2\pi nt}{L}\right) \right]
\]

where \( F(t) = \int_{0}^{t} f(s) \, ds \) is the antiderivative of \( f \) with \( F(0) = 0 \).

Where (16) should be read: \( F(t) - \frac{a_0 t}{2} \) is a \( P \) periodic function whose F.S.

\[
\hat{F}(k) \sim \sum_{n=1}^{\infty} \left[ -\frac{b_n}{\pi n} \cos\left(\frac{2\pi nk}{L}\right) + \frac{a_n}{\pi n} \sin\left(\frac{2\pi nk}{L}\right) \right]
\]

One can put equal sign in (16) because \( F(t) - \frac{a_0 t}{2} \) is piecewise \( C^1 \) and Convergence Theorem 2 applies.