3. Higher order, linear, diff. Eqn's

Consider the third order, linear, diff. eqn:

\[ a_3(x) y''' + a_2(x) y'' + a_1(x) y' + a_0(x) y = f(x) \]

or the n-th order, linear, diff. eqn:

\[ a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + ... + a_0(x) y = f(x) \]

where \( x \in I \), \( I \) an interval in \( \mathbb{R} \), \( f, a_0, a_1, ..., a_n \) continuous on \( I \), \( a_n(x) \neq 0 \) on \( I \).

(i) The homogeneous eqn: \( f = 0 \)

In this case all solv's of (1) are given by

\[ y = c_1 y_1 + c_2 y_2 + c_3 y_3 \], \( c_1, c_2, c_3 \in \mathbb{R} \)

where \( y_1, y_2, y_3 \) are three linear independent solv's of (1).
All solutions of (2) are given by:

\[ y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n, \quad c_1, c_2, \ldots, c_n \in \mathbb{R} \]

where \( y_1, y_2, \ldots, y_n \) are \( n \) linearly independent solutions of (2).

**Definition:** The functions \( y_1, y_2, y_3 \) are linearly dependent if there exist constants \( \alpha_1, \alpha_2, \alpha_3 \) not all zero such that:

\[ \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = 0 \]

Otherwise \( y_1, y_2, y_3 \) are called linearly independent.

The functions \( y_1, y_2, \ldots, y_n \) are linearly dependent if there exist constants \( \alpha_1, \alpha_2, \ldots, \alpha_n \) not all zero such that:

\[ \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n = 0 \]

Otherwise \( y_1, y_2, \ldots, y_n \) are called linearly independent.

**Example:** \( 1, x, 2x + 2 \) are linearly dependent because:

\[ 2 \cdot 1 + 2 \cdot x - 1 (2x + 2) = 0 \]

i.e. \( \alpha_1 = 2, \alpha_2 = 2, \alpha_3 = -1 \).
Example 2: $1, x, x^2$ are linearly independent, because if
\[ p(x) = a_1 \cdot 1 + a_2 \cdot x + a_3 \cdot x^2 = 0 \]
then $p(x)$ must be the zero polynomial (any number is a root) hence $a_1 = a_2 = a_3 = 0$. So there are no $a_1, a_2, a_3$ not all zero that make $a_1 \cdot 1 + a_2 \cdot x + a_3 \cdot x^2 = 0$.

Linear Dependence Criterion: $y_1, y_2, y_3$ are linearly dependent if and only if
\[ W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = 0 \]

$y_1, y_2, \ldots, y_n$ are linearly dependent if and only if
\[ W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = 0 \]
Example 1

\[
\begin{array}{ccc}
1 & x & 2x + 2 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
\end{array}
\]

\[ \Rightarrow 0 \]

So \(1, x, 2x + 2\) are linearly dependent

Example 2

\[
\begin{array}{ccc}
1 & x & x^2 \\
0 & 1 & 2x \\
0 & 0 & 2 \\
\end{array}
\]

\[ = 2 \neq 0 \]

So \(1, x, x^2\) are linearly independent

In general, it is very hard or impossible to find three linearly independent linear functions (or "vectors") for each (1) or "n" linearly independent linear functions for each (2). There are two exceptions:
Case I. Constant coeff.:

\[(1c) \quad a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0\]

where \(a_0, a_1, a_2, a_3 \in \mathbb{R}, \quad a_3 \neq 0,\) or more generally

\[(2c) \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = 0\]

where \(a_0, a_1, \ldots, a_n \in \mathbb{R}, \quad a_n \neq 0\)

Look for solutions in the form:

\[y(x) = e^{rx}\]

plug in and divide by \(e^{rx}\) to get the characteristic eqn which will determine \(r:\)

\[a_3 r^3 + a_2 r^2 + a_1 r + a_0 = 0\]

or in general:

\[a_n r^n + a_{n-1} r^{n-1} + \ldots + a_0 = 0\]
Example 3 \( 2y''' - 13y'' + 22y' - 8y = 0 \)

Characteristic eqn:

\[
2n^3 - 13n^2 + 22n - 8 = 0
\]

\[
\Rightarrow n_1 = \frac{1}{2}, \quad n_2 = 2, \quad n_3 = 4
\]

So \( y_1 = e^{\frac{1}{2}x}, \quad y_2 = e^{2x}, \quad y_3 = e^{4x} \). They are linearly independent because:

\[
W = \begin{vmatrix}
 e^{\frac{x}{2}} & e^{2x} & e^{4x} \\
\frac{1}{2} e^{\frac{x}{2}} & 2e^{2x} & 4e^{4x} \\
\frac{1}{4} e^{\frac{x}{2}} & 4e^{2x} & 16e^{4x}
\end{vmatrix}
\]

\[
= e^{\frac{x}{2}} \cdot e^{2x} \cdot e^{4x} \begin{vmatrix}
 1 & 1 & 1 \\
\frac{1}{2} & 2 & 4 \\
\frac{1}{4} & 4 & 16
\end{vmatrix}
\]

\[
= e^{6.5x} \cdot \begin{vmatrix}
 4 - 2 & (4 - \frac{1}{2}) & (2 - \frac{1}{2})
\end{vmatrix} \neq 0
\]

So all solutions are given by: \( y = C_1 e^{\frac{x}{2}} + C_2 e^{2x} + C_3 e^{4x} \)
Important note: To find the roots of a third degree polynomial (as eq (3) above requires) one needs to guess one of the roots. An educated guess is a rational root: \( \frac{p}{q} \)

where \( p \) divides the dominant coeff \( (2 \text{ in the case of eq (3)}) \) and \( q \) divides the free coeff \( (8 \text{ in the case of eq (3)}) \).

So \( \pm \frac{1}{2}, \pm 1, \pm 2, \pm 4, \pm 8 \) are good guesses for eq (3). Plug in and see that for example \( n_1 = \frac{1}{2} \) works. Now use the long division of polynomials to factor out \( n - \frac{1}{2} \) in (3).

\[
\begin{array}{c|ccccc}
\multicolumn{2}{c}{} & 2n^3 & -13n^2 & +22n & -8 \\
\hline
2n^2 & -n^2 & & & & \\
\hline
& & -12n^2 & +22n & -8 \\
& & -12n^2 & +6n & \\
\hline & & & 16n & -8 \\
& & & 16n & -8 \\
& & & & 0 \\
\end{array}
\]

So (3) becomes:

\[
(n - \frac{1}{2})(2n^2 - 12n + 16) = 0 \Rightarrow n_{2,3} = \frac{12 \pm \sqrt{144-128}}{4}
\]
In general, if the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation:

(a) are all real and distinct then:

$$y_1 = e^{\lambda_1 x}, \ y_2 = e^{\lambda_2 x}, \ldots, \ y_n = e^{\lambda_n x}$$

are $n$ linearly independent solutions. See Ex. 3

(b) $\lambda_1 = \lambda_2 = \ldots = \lambda_k \neq \lambda_{k+1} \ldots$ then

$$y_1 = e^{\lambda_1 x}, \ y_2 = xe^{\lambda_1 x}, \ldots, \ y_k = x^{k-1} e^{\lambda_1 x}$$

are $k$ linearly independent solutions and $y_{k+1}, y_{k+2}, \ldots$ can be found by using $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n$.

Example 4: $y''' - 4y'' + 5y' - 2y = 0$

Characteristic equation $\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$

$$\lambda_1, \lambda_2 = 1, \ \lambda_3 = 2$$

$\Rightarrow y_1 = e^x, \ y_2 = xe^x, \ y_3 = e^{2x}$ are linearly independent solutions. All solutions are:

$$y = c_1 e^x + c_2 xe^x + c_3 e^{2x}$$
(c) \( n_1 = a + bi \) (so \( n_2 = a - bi \)) then

\[ y_1 = e^{ax} \cos bx, \quad y_2 = e^{ax} \sin bx \]
are two linearly independent solutions. The others can be found by using \( n_3, n_4, \ldots, n_m \).

**Example 5** \( y^{(4)} - y'' + y' - y = 0 \)

Characteristic eqn: \( n^4 - n^2 + n - 1 = 0 \)

\( n_1 = 1, n_2 = i, n_3 = -i \)

\( \Rightarrow y_1 = e^x, \quad y_2 = \cos x, \quad y_3 = \sin x \quad \text{and all others} \)

One \( y = c_1 e^x + c_2 \cos x + c_3 \sin x \)

**Remarks** By combining (a), (b), and (c) one can always construct \( n \) linearly independent solutions from \( n \) roots of the characteristic eqn.

**Core II**: Eqn of Euler Type:

\[ a_3 x^3 y^{(4)} + a_2 x^2 y'' + a_1 x y' + a_0 y = 0, \quad x \neq 0 \]

or in general

\[ a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_0 y = 0, \quad x \neq 0 \]
use the substitution
\[ x = e^t \quad \text{for} \quad x > 0 \]
\[ x = e^{-t} \quad \text{for} \quad x < 0 \]
to reduce this problem to the case of constant coeff,

\underline{Example 6} \quad x^3 y''' - x^2 y'' + 2xy' - 2y = 0, \ x > 0

\[ x = e^t \quad \Rightarrow \quad t = \ln x \]

\[
y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}
\]

\[
y''' = \frac{d}{dx} \left( \frac{1}{x} \frac{dy}{dx} \right) = - \frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \left( \frac{d^2 y}{dt^2} \cdot \frac{1}{x} \right)
\]

\[
y'' = \frac{d}{dx} \left( \frac{1}{x^2} \frac{d^2 y}{dt^2} - \frac{1}{x} \frac{dy}{dt} \right) = - \frac{2}{x^3} \frac{d^2 y}{dt^2} +
\]

\[
+ \frac{1}{x^2} \frac{d^3 y}{dt^3} \cdot \frac{1}{x} + \frac{2}{x^3} \frac{dy}{dt} - \frac{1}{x^2} \frac{d^2 y}{dt^2} \cdot \frac{1}{x}
\]

\[
= \frac{1}{x^3} \left[ \frac{d^3 y}{dt^3} - 3 \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} \right]
\]

Plug in the eqn: \[ \frac{d^2 y}{dt^3} - 4 \frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} - 2y = 0 \]

By Ex 4 \[ y_1 = e^t \cdot x, \quad y_2 = t e^t \cdot x \ln x, \quad y_3 = e^{2t} \cdot x^2 \]
In all other cases you are not required
to find $y_1, y_2, \ldots, y_n$.

(ii) The Inhomogeneous Eqn: $f \neq 0$

The sol's of (i) are:

$$ y = c_1 y_1 + c_2 y_2 + c_3 y_3 + y_p, \quad c_1, c_2, c_3 \in \mathbb{R} $$

where $y_1, y_2, y_3$ are linearly indep sol's of the homogeneous eqn.

$y_p$ can be found by the method of variation of constants (parameters):

$$ y_p(x) = c_1(x) y_1(x) + c_2(x) y_2(x) + c_3(x) y_3(x) $$

and $c_1(x), c_2(x), c_3(x)$ can be found from:

$$ \begin{cases} 
  c_1' y_1 + c_2' y_2 + c_3' y_3 = 0 \\
  c_1' y_1' + c_2' y_2' + c_3' y_3' = 0 \\
  c_1' y_1'' + c_2' y_2'' + c_3' y_3'' = f(x)/a_3(x) 
\end{cases} $$
Example 6.1 \( x^3 y''' - xy'' + 2xy' - 2y = x \)

From Ex 6 we know:

\[
y = C_1 x + C_2 x \ln x + C_3 x^2 + y_p.
\]

\[
y_p = C_1(x)x + C_2 x \ln x + C_3 (x) x^2
\]

\[
\begin{align*}
    C_1^{'x} + C_2^{' x} \ln x + C_3^{' x^2} &= 0 \\
    C_1^{'1} + C_2^{' (1+ \ln x)} + C_3^{' 2x} &= 0 \\
    C_1^{0} + C_2^{' \ln x} + C_3^{' 2x} &= f(x)/x^3
\end{align*}
\]

\[
C_1^{'} = \left| \begin{array}{ccc}
    0 & x \ln x & x^2 \\
    1 & 1+ \ln x & 2x \\
    0 & 1/ \ln x & 2
\end{array} \right| = \frac{1}{x^2} \left[ x^2 \ln x - x^2 \right]
\]

\[
= \frac{1}{x} \left( \ln x - 1 \right)
\]

\[
\Rightarrow C_1(x) = \frac{1}{2} \ln^2 x - \ln x
\]
\[ C_2 \left( \frac{1}{x} \right) = -\ln x \]

\[ C_3 \left( \frac{1}{x} \right) = -\frac{1}{x} \]

So \( y_p = \left( \frac{1}{2} \ln^2 x - \ln x \right)x - x \ln^2 x - x \]
\[ = -\frac{1}{2} x \ln^2 x - x \ln x - x \]

and
\[ y = C_1 x + C_2 x \ln x + C_3 x^2 - \frac{1}{2} x \ln^2 x - x \ln x - x \]
In general, for eqn (2), the solutions are

\[ y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n + y_p \]

where \( y_p(x) = c_1(x) y_1 + c_2(x) y_2 + \ldots + c_n(x) y_n \)

and \( c_1(x), c_2(x), \ldots, c_n(x) \) are given by:

\[
\begin{aligned}
& c_1' y_1 + c_2' y_2 + \ldots + c_n' y_n = 0 \\
& c_1' y_1' + c_2' y_2' + \ldots + c_n' y_n' = 0 \\
& \quad \vdots \\
& c_1' y_1^{(n-1)} + c_2' y_2^{(n-1)} + \ldots + c_n' y_n^{(n-1)} = f(x)/a_{n}x^n(x)
\end{aligned}
\]
Finding $y_p$ for constant coeff. eqn's with special $f(x)$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = f(x)$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$ and

$f(x) = P(x) e^{\alpha x} \cos \beta x$ or $f(x) = P(x) e^{\alpha x} \sin \beta x$

where $P(x)$ is a polynomial then:

1° If $\lambda + i\beta$ is not a root of the characteristic eqn then:

$$y_p(x) = Q_1(x) e^{\alpha x} \cos \beta x + Q_2(x) e^{\alpha x} \sin \beta x$$

where $Q_1, Q_2$ are polynomials of the same degree as $P(x)$ but with undetermined coeff.

2° If $\lambda + i\beta = \eta_1, \eta_2, \ldots = \eta_k + \eta_k + 1$ where

$\eta_1, \eta_2, \ldots, \eta_k$ are the roots of the characteristic eqn:

$$y_p = x^2 \left[ Q_1(x) e^{\alpha x} \cos \beta x + Q_2(x) e^{\alpha x} \sin \beta x \right]$$

where $Q_1, Q_2$ are as in 1°

Remark: If $f(x) = \eta_0(x) + \eta_1(x) + \ldots + \eta_k(x)$ where each $\eta_i(x)$ is the form above then use the principle of superposition (see Lecture 13).
Example 4.1 \[ y''' - 4y'' + 5y' - 2y = e^x \]

From example 4 we know
\[ \lambda_1 = \lambda_2 = 1 \quad \lambda_3 = 2 \]

\[ e^x = P(x)e^{\lambda x} \cos \beta x \quad P(x) = 1, \quad \lambda = 1, \quad \beta = 0 \]

\[ \lambda + i\beta = 1 = \lambda_1 = \lambda_2 \Rightarrow 2^0 \text{ equals with } k = 2, \]

Q. & Q. are polynomials of degree zero:
\[ \Rightarrow \quad y_0 = x - A e^x \]

\[ y_1' = (2x + x^2) 4 e^x \]
\[ y_1'' = (2 + 4x + x^2) 4 e^x \]
\[ y_1''' = (6 + 6x + x^2) 4 e^x \]

Plug in the equation
\[ \Rightarrow \quad -2 A e^x = e^x \Rightarrow A = -\frac{1}{2} \]

\[ \Rightarrow \quad y_1' = \frac{x^2}{2} e^x \quad \text{and all slu's are:} \]
\[ y = c_1 e^x + c_2 x e^x + c_3 e^{2x} + \frac{x^2}{2} e^x \]
(iii) The initial value problem

Once all solutions of (1) respectively (2) have been found one can determine the solution constants from initial value data:

\[ y(b_0) = \tilde{y}_0, \quad y'(b_0) = \tilde{y}_1, \quad y''(b_0) = \tilde{y}_2 \]

for eq (1) respectively:

\[ y(t_0) = \tilde{y}_0, \quad y'(t_0) = \tilde{y}_1, \ldots, \quad y^{(n-1)}(t_0) = \tilde{y}_n \]

for eq (2).

Example 4.2 \( y''' - 4y'' + 5y' - 2y = e^x \)

with initial conditions:

\[ y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0 \]

We know from Ex 4.1 that the sol's are:

\[ y = C_1 e^x + C_2 x e^x + C_3 e^{2x} + \frac{x^2}{2} e^x \]
Plug in the initial conditions:

\[ 1 = y(0) = C_1 + C_3 \]
\[ 0 = y'(0) = C_1 + C_2 + 2C_3 \]
\[ 0 = y''(0) = C_1 + 2C_2 + 4C_3 + 1 \]

By solving this 3x3 system one gets:

\[ C_1 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix}} = \frac{1}{1} = 1 \]

\[ C_2 = \frac{\begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix}} = \frac{-1}{1} = -1 \]

\[ C_3 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix}} = 0 \]

So:

\[ y = e^x - xe^x + \frac{x^2}{2} e^x \]