Summary:


2. (iii) Applications of second order constant coeff. diff. eq.

1. (iv) Continuation: Principle of Superposition

(1) \[ a(x) y'' + b(x) y' + c(x) y = f_1(x) + f_2(x) \]
\[ a, b, c \text{ continuous, } a(x) \neq 0 \text{ for all } x. \]

Recall from Lecture 11: all soln's of (1) are given by:

\[ y = y_p + c_1 y_1 + c_2 y_2 \]
where \( y_1, y_2 \) are 2 linearly independent soln's of the homogeneous eq:

\[ a y'' + b y' + c y = 0 \]

and \( y_p \) is one particular soln of (1) that can be obtained by the method of Variation of Constants.
Theorem 4 (Principle of superposition)

A particular solution of the homogeneous eq (1):

\[ L[\Sigma Y_p] = f_1 + f_2 \]

can be obtained as:

\[ Y_p = Y_{p1} + Y_{p2} \]

where

\[ L[Y_{p1}] = f_1, \quad L[Y_{p2}] = f_2. \]

Proof: Let \( Y_{p1}, \ Y_{p2} \) be such that

\[ L[Y_{p1}] = f_1, \quad L[Y_{p2}] = f_2 \]

Then:

\[ L[Y_{p1} + Y_{p2}] = L[\Sigma Y_p] = \]

by linearity

\[ = f_1 + f_2 \]

So \( Y_{p1} + Y_{p2} \) solves (1)!

Generalized principle of superposition

A particular solution of

\[ L[Y] = f_1 + f_2 + \ldots + f_n \]

can be obtained as:

\[ Y_p = Y_{p1} + Y_{p2} + \ldots + Y_{pn} \text{ where } \]

\[ L[Y_{pk}] = f_k, \quad k = 1, 2, \ldots, n. \]
Example 5: \( y'' + y = e^x + \cos x \)

\[ y_p = y_{p_1} + y_{p_2} \text{ where} \]

\[ y''_{p_1} + y_{p_1} = e^x, \quad y''_{p_2} + y_{p_2} = \cos x \]

Use method of undetermined coeff:

- First compute roots of characteristic eq.

\[ n^2 + 1 = 0 \]
\[ n_{1,2} = \pm i \]

- Use Theorem 3:

\[ y_{p_1} = A e^x, \text{ plug in eq for } y_{p_1}: \]

\[ A e^x + A e^x = e^x \implies A = \frac{1}{2} \implies y_{p_1} = \frac{e^x}{2} \]

\[ y_{p_2} = x(A \cos x + B \sin x), \text{ plug in eq for } y_{p_2} \text{ use Leibnitz rule:} \]

\[ (f g)^{\prime\prime} = f^{\prime\prime} g + 2f^{\prime} g' + f g'' \]

\[ 2 (-4 \sin x + 3 \cos x) + x (-4 \cos x - 3 \sin x) + x (4 \cos x + 3 \sin x) = \cos x \]
\[-2A \sin x + 2B \cos x = \cos x\]
\[
\Rightarrow A = 0, \quad B = \frac{1}{2} \quad \Rightarrow \quad Y_p = \frac{1}{2} \sin x
\]

So,
\[
Y_p = \frac{e^x}{2} + \frac{x}{2} \sin x
\]

All solutions are:
\[
Y = \frac{e^x}{2} + \frac{x}{2} \sin x + C_1 \sin x + C_2 \cos x
\]

Suggested HW: Compare the method above which uses Principle of Superposition and the Method of Undetermined Coefficients with the Method of Variation of Constants or Example 5.
2 (iii) Applications of second order linear constant coeff equs.

(a) Ullon - Spring systems in Vertical

\[ F(t) \]

\[ m \]

\[ x = \text{displacement from equilibrium} \]
\[ m > 0 \]

\[ m x'' = -kx - cx' + F(t) \]

\[ k = \text{constant of elasticity of the spring, } k > 0 \]
\[ c = \text{friction constant, } c > 0 \]

Wheel - Coal - String System, Pendulum

\[ k = \frac{g}{l} \]
\[ c \approx 0 \]
(b) LRC circuits in Electricity

\[ E = U_L + U_R + U_C \]

Kirchhoff's Law:

\[ U_L = L \frac{dI}{dt} \quad L = \text{inductance} > 0 \]

\[ U_R = RI \quad R = \text{resistance} \geq 0 \]

\[ U_C = \frac{Q}{C} \quad C = \text{capacitance} > 0 \]

\[ Q = \text{charge} \Rightarrow \frac{dQ}{dt} = I \]

So

\[ L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E(t) \]
Mechanical - Electrical analogy:

\[ mx'' + cx' + kx = F(t) \]
\[ L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = E(t) \]

If one identifies:

\[ x \leftrightarrow Q \]
\[ m \leftrightarrow L \]
\[ c \leftrightarrow R \]
\[ k \leftrightarrow \frac{1}{C} \]
\[ F(t) \leftrightarrow E(t) \]

The systems are equivalent.
Classification based on homogeneous eq:

\[ mx'' + cx' + kx = 0 \quad m, k > 0, c > 0 \]

Characteristic eq:

\[ m\lambda^2 + c\lambda + k = 0 \]

1) Overdamped:

\[ c^2 - 4mk > 0 \]

\[ \Rightarrow \text{Roots} \ \lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} < 0 \]

\[ \Rightarrow \text{The soln of homogeneous eq}:
\]

\[ y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \]

\[ \text{decays exponentially} \]

2) Critically damped:

\[ c^2 = 4mk \]

\[ \Rightarrow \text{Roots} \ \lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -c/2m < 0 \]

\[ \Rightarrow \text{The soln of homogeneous eq}:
\]

\[ y = (C_1 + C_2 t) e^{-c/2m t} \]

\[ \text{decays like a polynomial} \]

\[ \text{terms are exponential} \]

3) Underdamped:

\[ c^2 < 4mk \]

\[ \Rightarrow \text{Roots} \ \lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \text{ are complex with negative real part} \]

\[ \Rightarrow \text{The soln of homogeneous eq}:
\]

\[ y = e^{-c/2m t} \left[ C_1 \cos \left( \sqrt{\frac{4mk - c^2}{2m}} t \right) + C_2 \sin \left( \sqrt{\frac{4mk - c^2}{2m}} t \right) \right] \]

\[ \text{decays but oscillates} \]
Undamped, $c = 0$. This is a particular case of $\text{(i)}$ but no decay in the Shen:

$$y = C_1 \cos\left(\sqrt{\frac{k}{m}} \ t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}} \ t\right)$$

Effects induced by the inhomogeneous forcing term

Undamped case:

$$m \frac{d^2 x}{dt^2} + k x = F \cos(\omega t)$$

Similar results hold for $\sin(\omega t)$ instead of $\cos(\omega t)$.

For $\sqrt{\frac{k}{m}} \neq \omega$,

$$x_p = \frac{F}{k - m\omega^2} \cos(\omega t)$$

and

$$x = C_1 \cos\left(\sqrt{\frac{k}{m}} \ t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}} \ t\right) + x_p$$

Bests $\omega$ close to $\omega_0 = \sqrt{\frac{k}{m}}$, $x(0) = x'(0) = 0$

$$x = \frac{F}{k - m\omega^2} \left(\omega \sin(\omega t - \omega_0 t) - \omega_0 \cos(\omega_0 t)\right) = \frac{F}{k - m\omega^2} \left((-2) \sin\left(\frac{\omega - \omega_0}{2} t\right) \sin\left(\frac{\omega + \omega_0}{2} t\right)\right)$$
Example \( m = 1, k = 4, F = 1, \omega = 1.6 \)

\[ \omega_0 = \sqrt{\frac{k}{m}} = 2 \] is close to \( \omega = 1.6 \) see the graph of

\[ x_T = \frac{2}{4-(1.6)^2} \sin(0.2t) \sin(1.8t) \]

Beats with \( \Omega = 1.6 \) and \( \Omega_0 = 2 \)
For \( \omega_0 = \sqrt{\frac{k}{m}} = \omega \Rightarrow \text{Resonance} \)

\[
X_p = \frac{F}{2m\omega_0} \sin \omega_0 t \text{ linearly growing amplitude!}
\]

\[X = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + X_p\]

\[\text{Damped Eq: } c > 0\]

\[m\ddot{x} + cx' + kx = F \omega_0 \cos(\omega_0 t)\]

Similar results hold for \( \sin(\omega_0 t) \) instead of \( \cos(\omega_0 t) \)

\[
X_p = \frac{Fc\omega}{(k-c\omega^2)^2 + c^2\omega^2} \sin(\omega_0 t) + \frac{F(k-c\omega^2)}{(k-c\omega^2)^2 + c^2\omega^2} \cos(\omega_0 t)
\]

Note: \( A \sin(\omega_0 t) + B \cos(\omega_0 t) = C \cos(\omega_0 t - \phi) \)

where \( C = \sqrt{A^2 + B^2} \)

and \( \tan \phi = \frac{A}{B} \)

\[
X_p = \frac{F}{\sqrt{(k-c\omega^2)^2 + c^2\omega^2}} \cos(\omega_0 t - \phi) \text{ where}
\]

\[\phi = \tan^{-1} \frac{c\omega}{k-\omega^2}\]
Practical Resonance: When the amplitude of the particular sin is maximized.

Example: \( m = 1, \ c = 2, \ k = 4, \ F = 1 \)