Summary:

1. Organizational issues
2. Differential eq and real world problems
3. Formal definition of diff equ's and soln's
4. Exact soln's of ODE's

1. Organizational issues:
   - First day handout
     - Test policies
   - IODE software
     - EWS Labs and Accounts
       (meet in GL next Tue)

2. Diff. eq's and Real World Problems:

Problem: A population of bacteria numbers 1,000 midindividuals. After 1 hour, the population doubles. What will
be the number of bacteria after $1\frac{1}{2}$ hours?

Answer:

**Step 1:** $P(t) =$ # of bacteria at time $t$ hours.

- $P(0) = 1,000$
- $P(1) = 2,000$

$\beta(t) =$ # of births per unit of population per unit of time

$\delta(t) =$ # of deaths per unit of population per unit of time

$$P(t + \Delta t) - P(t) = \text{births} - \text{deaths}$$

$$\leq \beta(t) P(t) \Delta t - \delta(t) P(t) \Delta t$$

$$\frac{P(t + \Delta t) - P(t)}{\Delta t} \leq [\beta(t) - \delta(t)] P(t)$$

$$\lim_{\Delta t \to 0} \Rightarrow \frac{dP}{dt} (t) = [\beta(t) - \delta(t)] P(t)$$
Assumptions: (see Sect. 2.1 for generalization)
\[ \beta(t) = \text{constant} = \beta \]
\[ \delta(t) = \text{constant} = \delta \]

Solution: \( \beta - \delta = k \)

\[ \frac{dp}{dt} = kp \quad k = \text{constant} \]

Step 2: All steps of the above equations are (see Lecture 2)

\[ p(t) = Ce^{kt} \quad C = \text{constant} \]

Use \( p(0) = 1,000 \), \( p(1) = 2,000 \) to determine \( C \) and \( k \)

\[ p(0) = 1,000 \implies C = 1,000 \]
\[ p(1) = 2,000 \implies 1,000 e^{k} = 2,000 \implies k = \ln 2 \]

\[ \implies p(t) = 1,000 (e^{\ln 2})^t = 1,000 \cdot 2^t \]

Step 3: \( p(1.5) = 1,000 \cdot 2^{1.5} \approx 2828 \)
So 2828 bacteria at 1 1/2 hours.
Real world situation → Step 3: Interpretation

Step 1: Formulation & assumptions

Mathematical model → Step 2: Analysis

Mathematical results

Improved model

Remark: Lectures focus on Step 2 = Mathematical Analysis = finding solutions on approximations or properties of solutions. Textbook adds on Step 1 = Modelisation and Step 3 = Interpretation.
3. Formal definitions:

**Differential Eqn**: An equation that involves an unknown function and its derivatives.

Example 1: \( p'(t) = k \ p(t) \quad k = \ln 2 \)

Example 2: \( y''(t) + ky(t) = 0 \quad k > 0 \)

Example 3: \( y'' = -g \quad g = 9.8 \text{ m/sec}^2 \)

In general:

\[
\begin{cases}
    F(t, y(t), y'(t), \ldots, y^{(n)}(t)) = 0 \\
    t \in I \subseteq \mathbb{R} \text{ an interval} \\
    y(t_0) \in I_0, y'(t) \in I_1, \ldots, y^{(n)}(t) \in I_n \text{ when } t \in I \\
    \text{and } I_0, I_1, \ldots, I_n \text{ are intervals}
\end{cases}
\]

\( n = \text{the order of the eqn} \)

Ex 1 is first order, Ex 2 & 3 are second order!
**Definition of Solution:** A continuous function $y: J \rightarrow \mathbb{R}$, $J \subseteq I$ are interval, such that $y(t), y'(t), \ldots, y^{(n)}(t)$ exist for $t \in J$ and $y(b) \in I_1$, $y'(b) \in I_2$, $\ldots$, $y^{(n)}(b) \in I_n$ when $t = b$ and

$$f(t, y(t), y'(t), \ldots, y^{(n)}(t)) = 0$$

is true for all $t \in J$ is called a solution of the ordinary differential eq (1).

**Example 1**
\[ k = \ln 2, \quad P(t) = 1000 \cdot 2^t \]
\[ P(t) = \sin \pi t \]
\[ y(t) = \cos(\sqrt{k} t) \]

**Example 2**
\[ y(t) = C_1 \cos(\sqrt{k} t) + C_2 \sin(\sqrt{k} t) \]

**Definition of General Solution:** The solution depending on "n" arbitrary constants where "n" is the order of the eqn.

**Example 1**
\[ P: [0, \infty) \rightarrow \mathbb{R}, \quad P(t) = 1000 \cdot 2^t \]

**Example 2**
\[ y: \mathbb{R} \rightarrow \mathbb{R}, \quad y(t) = C_1 \cos(\sqrt{k} t) + C_2 \sin(\sqrt{k} t) \]
Definition of a singular solution: A solution that cannot be obtained from the general solution.

Examples 1, 2 & 3 have no singular solutions. We will encounter them in Lecture 2.

General SLL, singular SLL's and SLL's that are a combination of the two form the set of all SLL's of an ODE.

4. Exact Solutions

(i) \( y'(t) = f(t) \) if given

Example 4: \( y' = -9.8 t \Rightarrow \)

\[
\Rightarrow y(t) = \int -9.8 t \, dt + C = -\frac{9.8}{2} t^2 + C
\]

This is the general SLL of Ex 4

Since it depends on one arbitrary constant
Are there any other (singular) solutions? NO, because any two solutions would have the same derivatives \( n \) times from calculus I. They can only differ by a constant.

Generalization: The solutions of

\[ y^{(n)}(t) = f(t) \]

can be calculated recursively:

\[ y^{(n-1)}(t) = \int f(s) \, ds + C_{n-1} = f_1(t) + C_1 \]

\[ y^{(n-2)}(t) = \int f_1(s) \, ds + C_{n-2} t + C_2 = f_2(t) + C_1 t + C_2 \]

\[ \vdots \]

\[ y(t) = \int f_{n-1}(s) \, ds + C_{n-1} t^{n-1} + C_{n-2} t^{n-2} + \ldots + C_n \]
Example 3 \[ y'' = -g \quad y = 9.8 \text{ m/s}^2 \]
\[ y' = -gt + c_1 \]
\[ y = -\frac{g}{2} t^2 + c_1 t + c_2 \] \(\ast\)

Remarks
This is vertical motion
in a gravitational field!
In practice, \(c_1, c_2\) can
be determined from the initial
height and velocity:

\[ y(0) = y_0 = \text{given \# say } 2 \]
\[ y'(0) = y_1 = \text{given \# say } 0 \]

Plug in into \(\ast\)

\[ 2 = y(0) = -\frac{g}{2} 0^2 + c_0 + c_2 = c_2 \]
\[ \Rightarrow c_2 = 2 \]

\[ 0 = y'(0) = -gt + c_1 \bigg|_{t=0} = -g \cdot 0 + c_1 \]
\[ \Rightarrow c_1 = 0 \]

Hence \[ y(t) = -\frac{g}{2} t^2 + 2 \]