The purpose of this handout is to show you that Euler method converges to the exact solution and to propose a few related, optional homework problems. If you solve both of them carefully and correctly by Thursday, October 1, you get extra credit for the midterm.

In principle, we say that a numerical method converges to the exact solution if decreasing the step size leads to decreased errors such that in the limit when the step size goes to zero the errors go to zero. All numerical methods you encountered so far have this property when applied to well behaved equations (the ones for which the right hand side, \( f(t, x) \), has continuous partial derivatives in a region around the initial data. In particular the Existence and Uniqueness Theorem must apply to these problem). Here is the argument for Euler Method. Consider the differential equation:

\[
\begin{aligned}
&x'(t) = f(t, x) \\
x(t_0) = x_0 
\end{aligned}
\]  

and the Euler approximations with step size \( h > 0 \):

\[
\begin{aligned}
t_{i+1} &= t_i + h \\
x_{i+1} &= x_i + hf(t_i, x_i)
\end{aligned}
\]

which starts with \( i = 0 \) and inductively finds approximations \( x_1, x_2, \ldots, x_n \) for the values of the solution at inputs \( t_1, t_2, \ldots, t_n \). We break the argument into two parts:

\section{Part I: Local error and local truncation error}

Local error = error after the first step = \( x(t_0 + h) - x_1 \)

Taylor series for the exact solution give:

\[
\begin{aligned}
x(t_0 + h) &= x(t_0) + x'(t_0)h + x''(\tilde{t})h^2/2 \\
&= x_0 + f(t_0, x_0)h + x''(\tilde{t})h^2/2 
\end{aligned}
\]

where \( \tilde{t} \) is between \( t_0 \) and \( t_0 + h \).

\textbf{Remark 1} For other numerical schemes one may need to use higher order Taylor series like:

\[
x(t_0 + h) = x(t_0) + x'(t_0)h + x''(t_0)h^2/2 + x'''(\tilde{t})h^3/6.
\]
Subtracting now (3) from the first step in Euler method, that is (2) with \( i = 0 \), we get:

\[
x(t_0 + h) - x_1 = x''(\tilde{t})h^2/2
\]

hence

\[
|\text{local error}| \leq Ch^2 \quad (5)
\]

**Remark 2** It is not important at this stage but to estimate \( C \) for unknown exact solutions one differentiates (1) and by the chain rule:

\[
x''(\tilde{t}) = \frac{\partial f}{\partial t}(\tilde{t}, x(\tilde{t})) + \frac{\partial f}{\partial x}(\tilde{t}, x(\tilde{t}))x'(\tilde{t}) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} f.
\]

So, \( C \) is controlled by bounds on \( f \) and the gradient of \( f \).

By definition:

\[
\text{local truncation error} = LTE = \frac{\text{local error}}{\text{stepsize}}
\]

therefore:

\[
|LTE| \leq Ch \quad (6)
\]

# 2 Part II. Cumulative error

**error at step \( n \) = \( E_n = x(t_n) - x_n \).**

By Part I:

\[
x(t_n) = \text{Euler method starting at } x(t_{n-1}) + h \cdot LTE
\]

\[
= x(t_{n-1}) + hf(t_{n-1}, x(t_{n-1})) + h \cdot LTE
\]

By Euler method (2) at step \( i=n-1 \):

\[
x_n = x_{n-1} + hf(t_{n-1}, x_{n-1})
\]

Subtracting the two relations we have:

\[
E_n = E_{n-1} + h[f(t_{n-1}, x(t_{n-1})) - f(t_{n-1}, x_{n-1})] + h \cdot LTE. \quad (7)
\]

To estimate the difference in the square brackets we use again Taylor series in the second variable for \( f \):

\[
f(t_{n-1}, x(t_{n-1})) = f(t_{n-1}, x_{n-1}) + \frac{\partial f}{\partial x}(t_{n-1}, \tilde{x}) \frac{1}{E_{n-1}}(x(t_{n-1}) - x_{n-1})
\]

Replacing this in (7) we get:

\[
E_n = E_{n-1}(1 + h\frac{\partial f}{\partial x}) + h \cdot LTE
\]

hence:

\[
|E_n| \leq |E_{n-1}|(1 + hK) + h \cdot |LTE| \quad (8)
\]

where I assume that a uniform bound \(|\frac{\partial f}{\partial x}(t, x)| \leq K\) can be found for all \((t, x)\) in a rectangle containing both the solution and its approximations.
Optional Homework 1  Show that (8) implies:

$$|E_n| \leq |E_0|(1 + hK)^n + h \cdot |LTE| \cdot [1 + (1 + hK) + (1 + hK)^2 + \ldots + (1 + hK)^{n-1}]$$

Note that $E_0 = 0$ since the numerical methods starts from the exact initial data. Moreover, the sum in the square brackets is the sum of a geometric series: $1 + a + \ldots a^{n-1} = (a^n - 1)/(a - 1)$. The above relation for $E_n$ becomes:

$$|E_n| \leq |LTE|(1 + hK)^n - 1$$

(9)

The only difficult term left is $(1 + hK)^n$. Denote:

$$T = nh.$$  

(10)

$T$ is actually the time elapsed from the initial time to the last step. You might remember from Calculus that for $a > 0$, the sequence $(1 + a/n)^n$, $n$ natural number, is an increasing sequence convergent to $e^a$. We can write:

$$(1 + hK)^n = (1 + KT/n)^n \leq e^{KT}, \text{ for any natural number } n.$$  

Replacing now in (9) we get:

$$|E_n| \leq |LTE|e^{KT} - 1$$

(11)

From Part I, see (6), $|LTE| \leq Ch$, hence:

$$|E_n| \leq Ct h$$

(12)

where $C_T$ depends on $T$, the time elapsed from the initial time until the last step, $C_T$ also depends on the size of $f$ and its first partial derivatives, but it DOES NOT depend on the step size, $h$, or on how many steps, $n$, you need to reach the time $t_0 + T$. In conclusion we showed:

**Theorem 1** Consider the initial value problem (1) and the Euler numerical method (2). Consider the numbers $M$, $C$, $T > 0$ such that:

$$|f(t, x)| \leq M$$

$$|\frac{\partial f}{\partial t}(t, x)| \leq C$$

$$|\frac{\partial f}{\partial x}(t, x)| \leq K$$

for all $(t, x)$ in the rectangle $t_0 \leq t \leq t_0 + T$, $x_0 - MT \leq x \leq x_0 + MT$.

Then the errors between the exact solution $x(t)$ of (1) and the Euler approximations $x_i$ at $t_i$, $i = 0, 1, 2 \ldots n$ on the interval $[t_0, t_0 + T]$ are proportional to the step size $h$ and go to zero as $h \to 0$. More precisely:

$$|x(t_i) - x_i| \leq (C + KM)\frac{e^{KT} - 1}{2K} h$$

for all $i = 0, 1, 2, \ldots n$ with $n \leq \frac{T}{h}$. 

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Note that the numbers $M, C, T > 0$ with the properties required by the theorem above can always be found provided $f(t, x)$ has continuous partial derivatives in a region (rectangle) around the initial data $(t_0, x_0)$.

**Optional Homework 2** For the numerical method “improved Euler” do the following:

(i) Find its local error and local truncation error. Hint: Use the technique in Part I but this time rely on (4).

(ii) Calculate the cumulative error after $n$ steps. Compare with (11) and (12). What happens when the step size $h$ goes to zero.

### 3 Other types of errors

**Round off errors** The theorem above is true provided the arithmetic in calculating the numerical approximation is perfect. That is: when doing the operations required by (2) no errors occur. However computers always round off real numbers. For example in Matlab, and Fortran which are programming languages frequently used in science, all the numbers are rounded to the nearest eighth decimal (in single precision) or to the nearest sixteenth decimal (in double precision). In numerical methods rounding errors become important when the step size $h$ is comparable with the precision of the computations. Thus, running Euler method with $h \approx 10^{-8}$ may give worse approximation than running it with $h = 10^{-6}$ (in single precision), solely because of rounding errors.

**Under sampling** You encountered this phenomenon in Problem IV, Homework 5. It occurs when the step size of the numerical method is too large to capture the rapid changes in the slope field. While in principle this can be fixed by decreasing the step size it may turn up to be impractical to do so because:

- you may reach the round off threshold (see above),
- the number of steps necessary to reach the value of the independent variable that you are interested in becomes so large (see formula (10)) that the time it takes the computer to finish is prohibitively long.