Summary

§ 2.1 Population Models

§ 2.4 Numerical Methods

§ 2.1 Population Models

As in lecture 1:

\[ P(t) = \text{population at time } t \]

\[ P(t + \Delta t) - P(t) = (\text{# births between } t \text{ and } t + \Delta t) - (\text{# of deaths between } t \text{ and } t + \Delta t) \]

\[ \approx \beta(t, P(t)) P(t) \Delta t - \delta(t, P(t)) P(t) \Delta t \]

where

\[ \beta(t, P) = \text{birth rate at time } t \text{ when population is } P \]
Birth rate = \# of births / unit time / unit population

\[ \delta(t, \rho) = \text{death rate at time } t \text{ when population is } \rho \text{ where} \]

Death rate = \# of deaths / unit time / unit population

So:

\[ \lim_{\Delta t \to 0} \frac{P(t + \Delta t) - P(t)}{\Delta t} = (\beta(t, \rho) - \delta(t, \rho)) P(t) \]

\[ = \] \[ P'(t) = [\beta(t, \rho(t)) - \delta(t, \rho(t))] P(t) \] (1)

Remarks: (1) is called the general population model.

Specific models:

1° Exponential model: \( \beta(t, \rho) = \beta_0 \), \( \delta(t, \rho) = \delta_0 \)

\[ P'(t) = (\beta_0 - \delta_0)P(t) = kP(t) \]

\( P(0) = P_0 \)
First order linear equation with solution

\[ P(t) = P_0 e^{kt} \]

\[ = \begin{cases} \text{growth if } k = \beta_0 - \delta_0 > 0 \\ \text{decay if } k = \beta_0 - \delta_0 < 0 \end{cases} \]

and \( P(t) = P_0 \) if \( k = \beta_0 - \delta_0 = 0 \)

2° Logistic Equation:

\[ \beta(t) = \beta_0 - \beta_1 P, \beta_0, \beta_1 > 0 \]

i.e. the birth rate is linearly decreasing in population size (due for example to limited food supply), \[ \beta(t) = \delta_0 \]

(1) becomes

\[ P'(t) = aP - bP^2 \]  

where \( a = (\beta_0 - \delta_0) \) and \( b = \beta_1 \)

Remark: If both \( a > 0, b > 0 \) (2) is called the logistic equation.
(2) can be rewritten:

\[ p'(t) = kP(M-P) \]  

where \( k = b, M = \frac{a}{b} > 0 \) for logistic equation

(b) Equation (3) is also the final model if \( b = b_0 \) and \( b(0, P) = b_0 P \) in the general model (1), due for example to certain realistic behaviors, or competition

\[ \Rightarrow p'(t) = kP(M-P), \quad k = 2 > 0, \quad M = \frac{b_0}{2} > 0 \]

(c) \( M = \) total population, \( P = \) infected population. Rate of new infections is proportional to \# of scissorsets between infected and healthy individuals \( \leq P(M-P) \)

\[ \Rightarrow p'(t) = kP(M-P), \quad k, M > 0 \]
3° Doomsday / Extinction Models

\( P(t) = 12 P \quad k > 0 \quad \text{because} \)

# births/unit time proportional to # of

encounters between males and females \( \approx \frac{P^2}{2} \)

\( \sigma (t, P) = \sigma_0 > 0 \)

\( \Rightarrow \) (1) becomes

\( P' = (-k) P (P - M) \quad -k < 0, \quad M = \frac{\sigma_0}{k} > 0 \)

Conclusion: Equation (3)

\( P' = k P (P - M), \quad M > 0 \) (3)

models population dynamics in both logistic case \( k > 0 \), and doomsday/extinction case \( k < 0 \).

It is a separable equation and all solutions can be found by integration.
However the more features of the population
dynamics can be inferred from:

Bifurcation diagram for (3) with \( \mu > 0 \) fixed

\[
\begin{array}{cc}
\uparrow & \uparrow \\
\downarrow & \downarrow \\
O & O \\
\uparrow & \uparrow \\
\downarrow & \downarrow \\
M & M \\
\end{array}
\]

i.e. \( O, M \) are always equilibria with

\( O \) stable, \( M \) unstable for \( k < 0 \)

\( O \) unstable, \( M \) stable for \( k > 0 \)

So:
2° (in logistic case) $k > 0$ any initial population $N_0 > 0$ will evolve towards the limiting population $M$, which is sometimes called carrying capacity.

3° (in doomsday/extinction case) $k < 0$ an initial population $0 < N_0 < M$ will become extinct ($\lim_{t \to \infty} N(t) = 0$) unless an initial population $N_0 > M$ will blow up. $M$ is called threshold population as it separates extinction from blow-ups.

Blow-up is in fact a doomsday scenario as the population reaches infinity in finite time. This cannot be deduced from the bifurcation diagram, it requires solving

\[
\begin{align*}
    P(t) &= k P(M - P) \\
    P(t_0) &= N_0
\end{align*}
\]

with $k < 0$, $M > 0$, $N_0 > M$. 

\[
\int \frac{dv}{v(v-\mu)} = \int k dt \\
\ln \frac{v}{v-\mu} = kmt + C' \\
\frac{v}{v-\mu} = C_1 e^{kmt} \\
\frac{v}{\mu} = C_1 e^{kmt} \\
C_1 = \frac{v_0}{\mu} e^{-kmt} \\
\frac{v}{\mu} = \frac{v_0 e^{kmt}}{\mu} \\
\frac{v}{\mu} = \frac{v_0 e^{kmt}}{v_0 - \mu} \\
\Rightarrow v = \frac{v_0 \mu e^{kmt}}{v_0 e^{kmt} - (v_0 - \mu)} \\
The denominator becomes zero after time: \\
t - t_0 = \frac{1}{k\mu} \ln \left( \frac{v_0 - \mu}{v_0} \right).
\]
§ 2.4 Numerical Methods

These are algorithms used in computers to approximate solutions of ODE's.

A description of two numerical methods follows:

Euler Method

\[ x' = f(t, x) \]

You have an approx value \( x_0 \) at \( t_0 \)
(on initial condition). You want a value at \( t_0 + h \), \( h = \) step size:

\[ x'(t_0) \approx f(t_0, x_0) \quad \text{approximate slope} \]

\[ x(t_0 + h) \approx x(t_0) + h \times x'(t_0) = x_0 + h f(t_0, x_0) \]

\[ \approx \quad \text{approximate value at } t_0 + h \]
Graphical representation of Euler's method

\[ y' = y(3 - y) \]
Improved Euler Method

\[ x' = f(t, x) \]

You have an approx value \( x_0 \) at \( t_0 \) (or initial condition). You want an approx value at \( t_0 + h \), \( h = \text{step size} \):

\[ x'(t_0) \approx f(t_0, x_0) \quad \text{predicted slope} \]

\[ x_1 = x_0 + h f(t_0, x_0) \quad \text{intermediate point} \]

\[ x'(t_0 + h) = f(t_0 + h, x_1) \]

\[ x'(t_0) = \frac{f(t_0, x_0) + f(t_0 + h, x_1)}{2} \quad \text{corrected slope} \]

\[ x(t_0 + h) = x_0 + h \frac{f(t_0, x_0) + f(t_0 + h, x_1)}{2} \quad \text{predicted value at } t_0 + h \]
Graphical representation of Improved Euler Method

\[ y' = y(3 - y) \]

Legend: 
- Euler Step (Intermediate point)
- Improved Euler Step