Summary

4. Exact solutions for ODE's (cont)

(i) Exact diff eqns

(ii) Second order reducible eqns

1° Eq of type \( F(x, y', y'') = 0 \)

2° Eq of type \( F(y, y', y'') = 0 \)
4. Exact solutions of ODE's (continuation).

(vi) Exact Eqn

\[ \frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)} \]

Find the solution in the implicit form:

\[ F(x, y) = c \]

How do we find \( F \)? Along the's of the diff eqn:

\[ F(x, y(x)) = c \]

\[ \frac{d}{dx} F(x, y(x)) = 0 \]

Chain Rule:

\[ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0 \]
\[
\frac{dy}{dx} = -\frac{\frac{\partial^2 F}{\partial x \partial y}}{\frac{\partial^2 F}{\partial y}}
\]

One possibility is

\[
\begin{align*}
\frac{\partial F}{\partial x} &= P(x, y) \\
\frac{\partial F}{\partial y} &= Q(x, y)
\end{align*}
\]

From calculus III, for smooth functions,

\[
\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}
\]

compatible condition

If compatibility condition is satisfied, one uses calculus in to solve \((x, y)\) and get the general solution in the form

\[ F(x, y) = C, \quad C \in \mathbb{R} \]
If compatibility condition is not satisfied then \((\varphi)\) does not imply \((\varphi')\) but

\[
\frac{\partial \varphi}{\partial x} = p(x, y) S(x, y)
\]

\[
\frac{\partial \varphi}{\partial y} = q(x, y) S(x, y)
\]

Such that

\[
\frac{\partial}{\partial y} (p \varphi) = \frac{\partial}{\partial x} (q \varphi)
\]

\(S\) is called an integrating factor and is usually very hard to find. There are special situations in which one can easily construct \(S\), see book by:

Kamke, E: Differentialgleichungen I, 1977

Polyanin, A. D. and Zaitsev, V. F. Handbook of

exact sln's for ODE's, 2003

We will focus on the exact equation

\(\varphi\), in other words, \(S\) is not needed!
Example 12

\[
\frac{dy}{dx} = \frac{x^2 + y^2}{-2xy + 1}
\]

To check for exactness write it in the form:

\[
-(x^2 + y^2) \, dx + (-2xy + 1) \, dy = 0
\]

\[P(x, y) = -(x^2 + y^2), \quad Q(x, y) = -2xy + 1\]

\[
\frac{\partial}{\partial y} \left( -x^2 - y^2 \right) = -2y \quad \text{exact eqn}
\]

\[
\frac{\partial}{\partial x} \left( -2xy + 1 \right) = -2y
\]

Method I to find F:

\[
0 \frac{\partial F}{\partial x} = P(x, y) = \int P(x, y) \, dx + g(y)
\]

\[
= -\frac{x^3}{3} - y^2x + g(y)
\]
\[
\frac{\partial}{\partial y} F(x, y) = \varphi(x, y)
\]

\[\Rightarrow -2y + g'(y) = -2xy + 1 \quad (x)\]

\[\Rightarrow g'(y) = 1 \Rightarrow g(y) = y\]

\[
\Rightarrow F(x, y) = -\frac{x^3}{3} - y^2x + y
\]

The general solution is:

\[
F(x, y) = C
\]

\[-\frac{x^3}{3} - y^2x + y = C\]

**Method II to find F:**

Fix a point in the plane \((x_0, y_0)\). Integrate along any path connecting \((x_0, y_0)\) with \((x, y)\)
\[ F(x, y) = \int y \left( x^2 + y^2 \right) \, dx + \left( -2xy + 1 \right) \, dy \]

or in general

\[ F(x, y) = \int y P(x, y) \, dx + Q(x, y) \, dy \]

I'll choose \((x_0, y_0) = (0, 0)\) and the following simple path:

\[ F(x, y) = \int_0^x - \left( t^2 + 0 \right) \, dt + \int_0^y \left( 2xt + 1 \right) \, dt \]

\[ = -\frac{x^3}{3} - xy^2 + y \]
The solution of the problem is

\[-\frac{x^3}{3} - xy^2 + y = C\]

Remark: Method I relied on two integrations (first in \(x\), then in \(y\), the order can be reversed) and one cancellation, see (4) on page 6. It is valid only where the integrations never cross a point in which the cancellation foils, i.e., it is only valid on rectangles where:

- \(P(x, y)\), \(Q(x, y)\) are continuous and have continuous partial derivatives

- \(\frac{\partial P}{\partial y} (x, y) = \frac{\partial Q}{\partial x} (x, y)\)

Method II allows more flexibility in choosing the paths \(y\) (does not have to be the sides of a rectangle), hence it is valid in any simply connected set where the above two conditions are satisfied!
Example: \[ \frac{dy}{dx} = -\frac{\cos x + \ln y}{\frac{x}{y} + e^y} \] Write it as

\[(\cos x + \ln y)\,dx + \left(\frac{x}{y} + e^y\right)\,dy = 0\]

Check whether it is exact:
- Are partial derivatives of \(\cos x + \ln y\) and \(\frac{x}{y} + e^y\) continuous in a rectangle? Yes \(x \in \mathbb{R}, y > 0\)
- \(\frac{\partial}{\partial y} (\cos x + \ln y) = \frac{\partial}{\partial x} \left(\frac{x}{y} + e^y\right)\)

in the rectangle? Yes \(\frac{1}{y} = \frac{1}{y}\) on \(x \in \mathbb{R}, y > 0\)

Then all solutions (in the rectangle) are given by

\[ F(x, y) = C, \quad C \text{ arbitrary constant} \]

where \(\frac{\partial F}{\partial x} (x, y) = \cos x + \ln y; \frac{\partial F}{\partial y} (x, y) = \frac{x}{y} + e^y \)

\(F\) can be obtained by fixing \((x_0, y_0)\) in the rectangle and integrating along any path in the rectangle:
\[ F(x, y) = F(x_0, y_0) + \int_{y_0}^{y} (\cos x + \ln y) \, dx + \left( \frac{x}{y} + e^y \right) \, dy \]

where \( y \) connects \((x_0, y_0)\) with \((x, y)\)

\[ F(x, y) = 0 + \int_{0}^{\pi} (\cos t + \ln 1) \, dt + \int_{1}^{y} \left( \frac{x}{t} + e^t \right) \, dt \]

\[ = 0 + \sin t \bigg|_{0}^{\pi} + (x \ln t + e^t) \bigg|_{1}^{y} \]

\[ = 0 + \sin \pi + x \ln y + e^y - 2 \]

Hence

\[ \sin x + x \ln y + e^y = C \quad C \in \mathbb{R} \]

are all solutions \( x \in \mathbb{R}, \ y > 0 \) of the equation...
(vii) Second order reducible equations

1° \( F(x, y', y'') = 0 \)

Substitution

\( \nu(x) = y'(x) \Rightarrow \nu'(x) = y''(x) \)

The eqn becomes:

\( F(x, \nu, \nu') = 0 \)

First order eqn which may be solved by techniques (i) – (vii).

Example 13: \( y'' = \frac{y'}{x} \) \( x > 0 \)

\( \nu(x) = y'(x), \ \nu'(x) = y''(x) \)

\( \Rightarrow \nu' = \frac{\nu}{x} \)

which is separable and also first order linear.
2° $F(y, y', y'') = 0$

Look for $y' = p(y)$ $p$ an unknown function at this stage. By chain rule

$$y'' = p'(y)y' = p'(y)p(y)$$

So:

$$F(y, p, p'p) = 0$$

This is a first order eq in unknown function $p$ with independent variable $y$! Could be solved by (i)-(iv)

Once $p$ is known solve

$$y' = p(y)$$ separable!
Example 14. \( y y'' = -(y')^2 + y' y \)
\( y' = r(y), \ y'' = r'(y) y' = p' p \)

Substitute:
\( y p' p = -p^2 + ry \)

\( p = 0 \) is a solution. Otherwise, by dividing with \( p y \), \( y \neq 0 \), we get
\( p' + \frac{p}{y} = 1 \) first order linear.

\( g(y) = e^{\int \frac{1}{y} \, dy} = e^{|y|} = |y| \)

So \( (p |y|)' = |y| = \begin{cases} y & y > 0 \\ -y & y < 0 \end{cases} \)

\( \Rightarrow \ p |y| = \begin{cases} \frac{y^2}{2} + c & y > 0 \\ -\frac{y^2}{2} + c & y < 0 \end{cases} \)

\( \Rightarrow \ p = \frac{y^2}{2} + c |y| = \frac{y^2}{2} + \frac{c_1}{y} \)

On \( p \neq 0 \)
For $\rho = 0$ we get
\[ y' = 0 \Rightarrow y = C; \text{ } C \text{ arbitrary constant} \]

For $\rho = \frac{y^2 + c_1}{y}$ we get
\[ y' = \frac{y^2 + c_1}{y} = \frac{y^2 + c_1}{2y} \]
\[ \Rightarrow \int \frac{2y}{y^2 + c_1} \, dx = \int 1 \, dx \]
\[ \ln |y^2 + c_1| = x + c_2. \]
\[ \Rightarrow |y^2 + c_1| = e^{c_2} e^x \]
\[ \Rightarrow y^2 + c_1 = c_2 e^x \]
\[ \Rightarrow y = \pm \sqrt{c_2 e^x - c_1} \text{ as long as } c_2 e^x - c_1 > 0. \]

With the choice $c_2 = 0$, $\sqrt{-c_1} = C$, one can recover the $y = C$ solution obtained from $\rho = 0$. 

...
and \( y \neq 0 \) excluded when we cleared by \( y \).

So the solutions are:

\[
y = \pm \sqrt{c_2 e^x - c_1} \quad \text{with} \quad x \in \mathbb{R} \quad \text{if} \quad c_2 > 0, \ c_1 < 0
\]

\[
x > \ln \frac{c_1}{c_2} \quad \text{if} \quad c_1 > 0, \ c_2 > 0
\]

\[
x < \ln \frac{c_1}{c_2} \quad \text{if} \quad c_1 < 0, \ c_2 < 0
\]

It needs 2 initial conditions:

\[
y(x_0) = y_0, \ y'(x_0) = y_1
\]

For \( y_0 \neq 0, \ c_2, c_1 \) can be determined \( \Rightarrow \) unique sol.

For \( y_0 = 0, \ y_1 \neq 0 \) there are no solutions.

Check the sols.