Applications of linear equations to mechanical vibrations and electrical circuits.

The lecture combines sections 3.4, 3.6 and 3.7. The models reduce to second order, constant coefficient, linear equations

\[ ay''(t) + by'(t) + cy(t) = F(t) \]

where \( F(t) \) is a periodic function in \( t \).

We already know how to find all solutions of these type of equations. The lecture focuses on interpreting the solutions and discovering important phenomena exhibited by such systems:

- overdamping, critical damping, underdamping
- beats and resonance
(a) Mass - Spring systems in Mechanics

\[ x = \text{displacement from equilibrium} \]
\[ m = \text{mass} > 0 \]

Newton's Law: \[ m x'' = -kx - cx' + F(t) \]

- \( k \) = constant of elasticity of the spring, \( k > 0 \)
- \( c \) = friction constant, \( c > 0 \)

Wheel - Coal - String System, Pendulum

\[ F(t) \]
(b) LRC circuits in electricity

\[ E = U_L + U_R + U_C \]

\[ U_L = L \frac{dI}{dt} \quad L = \text{inductance} > 0 \]

\[ U_R = RI \quad R = \text{resistance} > 0 \]

\[ U_C = \frac{Q}{C} \quad C = \text{capacitance} > 0 \]

\[ Q = \text{charge} \implies \frac{dQ}{dt} = I \]

\[ L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E(t) \]
Mechanical - Electrical Analogy:

\[ mx'' + cx' + kx = F(t) \quad L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q(t) = E(t) \]

If one identifies:

\[ x \leftrightarrow Q \]
\[ m \leftrightarrow L \]
\[ c \leftrightarrow R \]
\[ k \leftrightarrow \frac{1}{C} \]
\[ F(t) \leftrightarrow E(t) \]

the systems are equivalent.

Classification based on the homogeneous eq.

\[ mx'' + cx' + kx = 0 \quad m, k > 0, c > 0 \]

Characteristic eq: \[ mr^2 + cr + k = 0 \]

The discriminant \[ \Delta = c^2 - 4mk \] determines
whether the roots are distinct, repeated or complex. Depending on the roots the solutions, hence the practical system, have different behavior:

**I. Overdamped:** $\Delta = c^2 - 4wmk > 0$

$(\Rightarrow)$ Roots $\lambda_1, \lambda_2$ of the characteristic eq

$$\lambda_{1,2} = \frac{-c \pm \sqrt{\Delta}}{2m}$$

distinct, real and negative

So, the solution of homogeneous eq:

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
decay exponentially

**II. Critically damped** $\Delta = c^2 - 4wmk = 0$

$(\Rightarrow)$ Roots $\lambda_1, \lambda_2$ of the characteristic eq are

$$\lambda_1 = \lambda_2 = \frac{-c}{2m}$$
Real, negative and repeated.

The solutions of the homogeneous eq:

\[ x(t) = (c_1 t + c_2 t) e^{-\frac{\xi}{2m} t} \]

decay like a polynomial times an exponential.

Underdamped, \( \Delta = c^2 - 4m\omega \leq 0 \)

the roots of the characteristic eqn are

\[ \xi_{1,2} = -\frac{c \pm \sqrt{\Delta}}{2m} \]

complex conjugate.

The solutions of the homogeneous eq:

\[ x(t) = c_1 e^{-\frac{\xi}{2m} t} \cos \left( \frac{\sqrt{\Delta}}{2m} t \right) + c_2 e^{-\frac{\xi}{2m} t} \sin \left( \frac{\sqrt{\Delta}}{2m} t \right) \]

decay exponentially but oscillate.
Using the formula:

\[
C_1 \cos \omega t + C_2 \sin \omega t = \sqrt{C_1^2 + C_2^2} \cos (\beta t - \phi)
\]

where \[\phi = \begin{cases} \tan^{-1} \left( \frac{C_2}{C_1} \right) & \text{if } C_1 > 0 \\ \pi + \tan^{-1} \left( \frac{C_2}{C_1} \right) & \text{if } C_1 < 0 \end{cases}\]

We get \[x(t) = \sqrt{C_1^2 + C_2^2} \, e^{-\frac{\alpha}{2m} t} \cos \left( \frac{\sqrt{-\Delta}}{2m} t - \phi \right)\]

So the maximum amplitude is \( \sqrt{C_1^2 + C_2^2} \) and it decays exponentially in time. The circular frequency of the oscillations are \( \omega_0 = \sqrt{-\Delta / (2m)} \).

**Remark.** The actual frequency (cycles/second) is \( \nu_0 = \frac{\omega_0}{2\pi} = \sqrt{-\Delta / (2\pi m)} \).

_**IV. Undamped** \( c = 0 \)

This is a particular case of _underdamped_

\[\nu_{1,2} = \pm i \frac{\sqrt{-\Delta}}{2m} = \pm i \sqrt{\frac{\nu_0}{2m}}\]

are purely imaginary.
The solutions of the homogeneous eq

\[ x(t) = C_1 \cos \left( \sqrt{\frac{k}{m}} \, t \right) + C_2 \sin \left( \sqrt{\frac{k}{m}} \, t \right) \]

oscillate forever. Using again (5) we get

\[ x(t) = \sqrt{c_1^2 + c_2^2} \cos \left( \omega_0 t - \phi \right) \]

where

\[ \omega_0 = \sqrt{\frac{k}{m}} \]

is called the natural circular frequency.

Remark: The actual natural (or internal) frequency is

\[ \nu = \frac{\omega_0}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \text{ cycles/second} \]

\( \sqrt{c_1^2 + c_2^2} \) is the amplitude of the oscillations.

\[ \phi = \frac{2}{\omega_0} \] is called the time lag (shift to the right of the graph of \( \cos \)).
Effects induced by the inhomogeneous (forcing) term

Bolts and resonances in Undamped system

\[ m x'' + k x = F \cos(\omega t) \]

Similar results hold for \( \sin(\omega t) \) instead of \( \cos(\omega t) \).

For \( \sqrt{\frac{k}{m}} \neq \omega \) we have by the method of undetermined coeff (check!)

\[ x_p = \frac{F}{k - mw^2} \cos(\omega t) \]

Hence

\[ x = C_1 \omega_0 \left( \sqrt{\frac{k}{m}} t \right) + C_2 \sin\left( \sqrt{\frac{k}{m}} t \right) + x_p \]

Bolts: \( \omega \) close to \( \omega_0 = \sqrt{\frac{k}{m}} \), \( x(0) = 0 = x'(0) \)

Then the solution has \( c_1 = -\frac{F}{\kappa - mw^2} \)
and can be rewritten:

\[ x = \frac{F}{k - m\omega^2} (\omega_0 \omega_0 t - \omega_0 \omega_0 t) \]

\[ = \frac{-2F}{k - m\omega^2} \sin\left(\frac{\omega - \omega_0 t}{2}\right) \sin\left(\frac{\omega + \omega_0 t}{2}\right) \]

slowly varying amplitude

So the graph looks like a highly oscillatory function with circular frequency close to \( \omega_0 \) but with a slowly modulated amplitude.

See next page for a concrete example.
Example: $m = 1, k = 4, F = 1, \omega = 1.6$

$\omega_0 = \sqrt{\frac{k}{m}} = 2$ is close to $\omega = 1.6$ see

the graph of

$$x_\nu = \frac{2}{4-(1.6)^2} \sin(0.2t) \sin(1.8t)$$

Beats with $\Omega = 1.6$ and $\Omega_0 = 2$
Resonance occurs when the frequency of the forcing term matches the natural (internal) frequency of the system:

$$\omega_0 = \sqrt{\frac{k}{m}} = \omega \implies \text{Resonance}$$

By the method of undetermined coefficients:

$$x_p = \frac{F}{2m\omega_0} \sin \omega_0 t$$

linearly growing amplitude!

$$x = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + x_p$$

Practical resonance in damped systems:

$$mx'' + cx' + kx = F \cos(\omega t), \quad c > 0$$

Similar results hold for \(F \sin(\omega t)\) forcing.

By the method of undetermined coefficients:

$$x_p = \frac{F\omega}{(k-m\omega^2) + c^2 \omega^2} \sin(\omega t) + \frac{(k-m\omega^2)F}{(k-m\omega^2)^2 + c^2 \omega^2} \cos(\omega t)$$
or using again (7)

\[ x_p = \frac{F}{\sqrt{(k-ww^2)^2 + c^2w^2}} \cos (\omega t - \phi) \]

where

\[ \phi = \begin{cases} \tan^{-1} \frac{cw}{k-ww^2} & \text{if } k-ww^2 > 0 \\ \pi + \tan^{-1} \frac{cw}{k-ww^2} & \text{if } k-ww^2 < 0 \end{cases} \]

Because the solution of the homogeneous equation decay exponentially, on the long run the system will relax to \( x_p \).

Depending on vehicle parameters (\( k, m, c, \omega \)) can be controlled in the system, choosing its value to obtain the maximum value of \( F/\sqrt{(k-ww^2)^2 + c^2w^2} \) puts the system in practical resonance.

Next page shows an example when the frequency of the forcing term are controlled:
Practical Resonance: When the amplitude of the particular sin is maximized.

Example: \( m = 1, \; c = 2, \; k = 4, \; F = 1 \)