6.1 Inner product, length, distance, orthogonality.

\[ \text{Def (Inner product)} \] If \( U, V \) are vectors in \( \mathbb{R}^n \):

\[
U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}
\]

then their inner product (dot product) is the number:

\[
U \cdot V = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n
\]

\[ \text{Examples} \]

\( 1^o \), \( U = [1, 2, 3]^T \), \( V = [2, 5, 6]^T \)

\[
U \cdot V = 1 \cdot 2 + 2 \cdot 5 + 3 \cdot 6 = 32
\]

\( 2^o \), \( 0 \cdot U = 0 \cdot u_1 + 0 \cdot u_2 + \ldots + 0 \cdot u_n = 0 \)
Remark \( U\cdot V = U^TV \)

product of two matrices

Properties of the inner product:

1. \( U\cdot U \geq 0 \) if and only if \( U = 0 \)
2. \( U\cdot V = V\cdot U \)
3. \( (U+V)\cdot W = U\cdot W + V\cdot W \)
4. \( C(U\cdot V) = (CU)\cdot V = U(CV) \) for \( C \) in \( \mathbb{R} \)

Def (length of a vector) If \( U \) is a vector in \( \mathbb{R}^n \), the length of \( U \) is the number:

\[ \|U\| = \sqrt{U\cdot U} \]

Remarks (i) If \( U = [u_1, u_2, \ldots, u_n]^T \) then

\[ \|U\| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2} \]

(ii) \( \|U\| = 0 \) if and only if \( U = 0 \)

Example \( \| [1, 2, 3]^T \| = \sqrt{14} \)
Definition (Distance between two vectors) If \( \mathbf{u}, \mathbf{v} \) are vectors in \( \mathbb{R}^n \) then the distance between them is the number:

\[
\text{dist} (\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||
\]

Remark: If \( \mathbf{u} = [u_1, u_2, \ldots, u_n]^T \) and \( \mathbf{v} = [v_1, v_2, \ldots, v_n]^T \) then

\[
\text{dist} (\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}
\]

Hence this notion generalizes the distance between points in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \)

Example:

\[
\text{dist} \left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right) = \sqrt{(-3)^2 + (-3)^2 + (-3)^2} = \sqrt{27}
\]

which is exactly the length of the segment with end-points at coordinates \( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) and \( \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \).
Def (orthogonal vectors) If \( \mathbf{u}, \mathbf{v} \) are vectors in \( \mathbb{R}^n \) they are called orthogonal (perpendicular) if
\[
\mathbf{u} \cdot \mathbf{v} = 0
\]

Examples \( [1, 2, 3]^T \) and \( [4, 5, 6]^T \) are not orthogonal because
\[
[1, 2, 3]^T \cdot [4, 5, 6]^T = 32 \neq 0
\]

\( [3, 1, 1]^T \) and \( [-1, 2, 1]^T \) are orthogonal because:
\[
[3, 1, 1]^T \cdot [-1, 2, 1]^T = -3 + 2 + 1 = 0
\]

Remark. The orthogonality of vectors generalizes the notion of perpendicular lines in plane or space. Indeed the following argument shows that:

\( \mathbf{u} \cdot \mathbf{v} = 0 \) if and only if the lines span \( \{ \mathbf{u} \} \perp \) span \( \{ \mathbf{v} \} \)

(the angle between the two lines is \( 90^\circ \)).
But
\[
\|u + v\|^2 = (u+v) \cdot (u+v) = u \cdot (u+v) + v \cdot (u+v)
\]
\[
= u \cdot u + u \cdot v + v \cdot u + v \cdot v
\]
\[
= \|u\|^2 + \|v\|^2 + 2u \cdot v
\]
\[
\|u - v\|^2 = (u-v) \cdot (u-v) = u \cdot (u-v) - v \cdot (u-v)
\]
\[
= u \cdot u - u \cdot v - v \cdot u + v \cdot v
\]
\[
= \|u\|^2 + \|v\|^2 - 2u \cdot v
\]
So
\[
\|u + v\| = \|u - v\| \iff \|u + v\|^2 = \|u - v\|^2 \iff
\]
\[
2u \cdot v = -2u \cdot v \iff u \cdot v = 0
\]

**Theorem (Pythagorean)** \( u, v \) are orthogonal if and only if:
\[
\|u + v\|^2 = \|u\|^2 + \|v\|^2
\]

**Proof** From (i) above
\[
\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2u \cdot v
\]
So
\[
\|u + v\|^2 = \|u\|^2 + \|v\|^2 \iff u \cdot v = 0.
\]