4.1 (Continuation) Linear combinations and spans
4.2 The Null space and Column space of a matrix

4.1. Linear combinations and spans

**Def** If $V$ is a vector space and $V_1, V_2, \ldots, V_p$ are in $V$ while $c_1, c_2, \ldots, c_p$ are real numbers then the vector:

\[
\left( (c_1 V_1 + c_2 V_2) + c_3 V_3 \right) + \ldots + (c_{p-1} V_{p-1}) + c_p V_p
\]

is called the linear combination of $V_1, V_2, \ldots, V_p$ with weights $c_1, c_2, \ldots, c_p$

Remark The parentheses in the formula above are not necessary because of property 2. of vector spaces which allows us to group the terms of the sum however we choose. From now on the linear combination will be denoted

\[
c_1 V_1 + c_2 V_2 + \ldots + c_p V_p
\]
Def. Let $V$ be a vector space and $v_1, v_2, \ldots, v_p$ in $V$ then

\[
\text{Span} \{ v_1, v_2, \ldots, v_p \}
\]

is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$.

Theorem. Let $V$ be a vector space and $v_1, v_2, \ldots, v_p$ in $V$. Then span $\{ v_1, v_2, \ldots, v_p \}$ is a subspace of $V$.

Proof. Check that span $\{ v_1, v_2, \ldots, v_p \}$ is closed under addition:

Let $u_1, u_2$ in span $\{ v_1, v_2, \ldots, v_p \}$ then there are weights $c_1, c_2, \ldots, c_p$ and $d_1, d_2, \ldots, d_p$ such that

\[
\begin{align*}
  u_1 &= c_1 v_1 + c_2 v_2 + \cdots + c_p v_p \\
  u_2 &= d_1 v_1 + d_2 v_2 + \cdots + d_p v_p
\end{align*}
\]

So

\[
\begin{align*}
  u_1 + u_2 &= (c_1 v_1 + c_2 v_2 + \cdots + c_p v_p) + (d_1 v_1 + d_2 v_2 + \cdots + d_p v_p) \\
  &= (c_1 + d_1) v_1 + (c_2 + d_2) v_2 + \cdots + (c_p + d_p) v_p
\end{align*}
\]

Change the order of addition (allowed by 4(i)) and regroup (4(ii)) until

\[
\begin{align*}
  u_1 + u_2 &= (c_1 v_1 + d_1 v_1) + (c_2 v_2 + d_2 v_2) + \cdots + (c_p v_p + d_p v_p) \\
  &= (c_1 + d_1) v_1 + (c_2 + d_2) v_2 + \cdots + (c_p + d_p) v_p
\end{align*}
\]

Use property 6.

\[
\begin{align*}
  u_1 + u_2 &= (c_1 + d_1) v_1 + (c_2 + d_2) v_2 + \cdots + (c_p + d_p) v_p
\end{align*}
\]
Hence $u_1 + u_2$ is a linear combination of $v_1, v_2, ..., v_p$ with weights $c_1 + d_1, c_2 + d_2, ..., c_p + d_p$ and (i) in definition of subspace (see Lecture 16) holds.

Check that span $\{u_1, u_2, ..., u_p\}$ is closed under multiplication by scalars:

Let $c$ in $\mathbb{R}$ and $u$ in span $\{u_1, u_2, ..., u_p\}$ then there are weights $c_1, c_2, ..., c_p$ such that

$$u = c_1 u_1 + c_2 u_2 + ... + c_p u_p$$

WTR

$$cu = c (c_1 u_1 + c_2 u_2 + ... + c_p u_p) = c c_1 u_1 + c c_2 u_2 + ... + c c_p u_p = (cc_1) u_1 + (cc_2) u_2 + ... + (cc_p) u_p$$

So $cu$ is a linear combination of $v_1, v_2, ..., v_p$ with weights $cc_1, cc_2, ..., cc_p$ and (ii) in definition of a subspace holds. Q.E.D.

Examples: the origin, lines through the origin, and planes through the origin are all subspaces of $\mathbb{R}^3$. In fact they are the only subspaces of $\mathbb{R}^3$ except $\mathbb{R}^3$ as we will see later.
4.2 The null space and column space of a matrix

Let \( A = [a_1 \ a_2 \ \ldots \ a_n] \) be a \( m \times n \) matrix.

**Def.** Null \( A \) is the set of all vectors \( x \in \mathbb{R}^m \) for which \( Ax = 0 \).

**Example** \( A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \)

\( \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \) is not in Null \( A \) because

\[
A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

\( \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \) is in Null \( A \) because

\[
A \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
We can find explicitly \( \text{Null } A \) by solving
\[
Ax = 0
\]
Use your reduction for \([A \ 0]\) and write the solution in the parametric vector form. For the example above see how
\[
\begin{bmatrix}
A & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 3 & 0 & 0 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6
\end{bmatrix}
\]
\[x_3 \text{ is free}
\]
\[
x_1 = 0, \ x_2 = 0, \ x_3 \text{ is free, } x_2 = -2x_3, \ x_1 = -3x_3
\]
\[
= \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
-3x_3 \\
-2x_3 \\
x_3 \\
0
\end{bmatrix} = x_3 \begin{bmatrix}
-3 \\
-2 \\
1 \\
0
\end{bmatrix} = \text{Span } \begin{bmatrix}
-3 \\
-2 \\
1 \\
0
\end{bmatrix}
\]

**Theorem:** If \( A \) is an \( m \times n \) matrix, then \( \text{Null } A \) is a subspace of \( \mathbb{R}^n \).

**Proof:** If \( x_1, x_2 \) are in \( \text{Null } A \) then
\[
A(x_1 + x_2) = Ax_1 + Ax_2 = 0 + 0 = 0 \text{ so } x_1 + x_2
\]
is in \( \text{Null } A \), (i) by definition of subspace closure.
If \( x_i \) is in \( \text{Null } A \) and \( c \) is in \( \mathbb{R} \), then
\[
A(cx_i) = cA(x_i) = c \cdot 0 = 0.
\]
So \( c \cdot x_i \) is in \( \text{Null } A \) and (ii) in the definition of a subspace checks too. (QED).

Remark. The columns of \( A \) are linearly independent if and only if \( \text{Null } A = \{ 0 \} \). See section 1.7 in your textbook.

**Def.** If \( A = [a_1, a_2, \ldots, a_m] \) is an \( m \times n \) matrix, then
\[
\text{col } A = \text{Span } \{ a_1, a_2, \ldots, a_m \}.
\]

Remark. \( \text{col } A \) is a subspace of \( \mathbb{R}^m \).

It is easy to find vectors in \( \text{col } A \). For example,
\[
A = \begin{bmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
then \( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \) is in \( \text{col } A \) or
\[
2 \cdot \begin{bmatrix}
3 \\ 2 \\ 0 \\ 0
\end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \\ 0 \end{bmatrix} \text{ is in } \text{col } A.
\]
To check whether a given vector \( \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) is in col A, one must find \( x \) in \( \mathbb{R}^n \) such that

\[
A x = \mathbf{b}.
\]

In our example,

\[
\begin{bmatrix}
A & \mathbf{b}
\end{bmatrix} \sim
\begin{bmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Thus pivot column

\( \Rightarrow A x = \mathbf{b} \) has at least one slu \( \Rightarrow \mathbf{b} \) is in col \( A \).

Remark \( \text{col} \ A = \mathbb{R}^m \) if and only if \( A \) has a pivot position in each row. See section 1.4 in your textbook.