2.2 (continuation) The inverse of a matrix.

**General properties of the inverse**

(i) If $A$ is invertible then $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$

(ii) If $A$ and $B$ are invertible then $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

(iii) If $A$ is invertible then $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$

Remark: The above properties come directly from the definition of invertible matrices and properties of products of matrices. For example for (ii):

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = B^{-1}B = I_n$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I_n$$

So $AB$ is invertible with inverse $B^{-1}A^{-1}$. 
Theorem. Let $A$ be an $n \times n$ matrix. $A$ is invertible if and only if $A$ is row-equivalent to $I_n$. Moreover, the inverse matrix can be calculated by row-reducing the augmented matrix $[A \mid I_n]$:

$$[A \mid I_n] \sim [I_n \mid A^{-1}]$$

Example 1. Check whether

$$A = \begin{bmatrix}
0 & 1 & 2 \\
1 & 0 & 3 \\
4 & -3 & 8
\end{bmatrix}$$

is invertible. If yes, find its inverse.

$$\begin{bmatrix}
0 & 1 & 2 & 0 & 0 \\
1 & 0 & 3 & 0 & 1 \\
4 & -3 & 8 & 0 & 0
\end{bmatrix} \overset{1^0}{\sim} \begin{bmatrix}
1 & 0 & 3 & 0 & 1 \\
0 & 1 & 2 & 1 & 0 \\
4 & -3 & 8 & 0 & 0
\end{bmatrix} \overset{2^0}{\sim} \begin{bmatrix}
1 & 0 & 3 & 0 & 1 \\
0 & 1 & 2 & 1 & 0 \\
0 & -3 & -4 & 0 & -4
\end{bmatrix}$$

$A^{-1} = I_3$
\[
\begin{bmatrix}
4 & 0 & 3 & 0 & 1 & 0 \\
0 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 1 & 3/2 & -2 & 1/2
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & -9/2 & 7 & -3/2 \\
0 & 1 & 0 & -2 & 4 & -1 \\
0 & 0 & 1 & 3/2 & -2 & 1/2
\end{bmatrix}
\]

So \( A \sim I_3 \Rightarrow A \) is invertible and

\[
A^{-1} = \begin{bmatrix}
-9/2 & 7 & -3/2 \\
-2 & 4 & -1 \\
3/2 & -2 & 1/2
\end{bmatrix}
\]

Above we used the following row operations:

1° Switch row 1 and 2
2° Subtract four times row 3 from row 1
3° Add to row 3 three times row 2
4° Divide row 3 by two
5° Subtract two times row 3 from row 2
6° Subtract three times row 3 from row 1.

Why does the algorithm work? To answer we need to understand elementary row operations:
Elementary matrices: There are all matrices that can be obtained from the identity matrix by performing one row operation.

**Examples**

\[
E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31}(4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}
\]

\(E_{12}\) is obtained from \(I_3\) by switching rows 1 and 2.

\(E_{31}(4)\) is obtained from \(I_3\) by adding 5 times row 3 to row 1.

Properties of elementary matrices:

a) If \(E\) is a \(n \times n\) elementary matrix and \(A\) is a \(n \times m\) matrix then \(EA\) is the matrix which is obtained from \(A\) by doing the same row operation as the one done to obtain \(E\) from \(I_n\).
Examples. For $A$ as in example 1

\[
A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}
\]

and $E_{12}$ the elementary matrix obtained from switching the first two rows:

\[
E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

we have

\[
E_{12}A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 4 & -3 & 8 \end{bmatrix} \text{ first two rows are switched.}
\]

The steps performed in example 1 to transform $A$ into $I_3$ can be rewritten as:

\[
E_{13}(-3)E_{23}(-2)E_3\left(\frac{1}{3}\right)E_{32}(3)E_{31}(-4)E_{12}A = I_3
\]

Try to identify the elementary matrices used above!
b) Elementary matrices are invertible

This is because any row operation can be undone. For example, to undo the switching of the first two rows you switch them again; so:

\[ E_{12} E_{12} = I_n \]

To undo adding to row 3 four times row 1 you subtract four times row 1 from row 3 four times row 1:

\[ E_{31} (-4) E_{31} (4) = I_n \]

Returning to example 1 we have

\[ \left( E_{12} (-3) E_{23} (-2) E_3 (\frac{1}{3}) E_{32} (3) E_{31} (-4) E_{12} \right) A = I_3 \]

inertible as a product of invertible matrices

\[ A = \left( E_{12} (-3) E_{23} (-2) E_3 (\frac{1}{3}) E_{32} (3) E_{31} (-4) E_{12} \right) I_3 \]

invertible as the inverse of an invertible matrix
So \( A \) is invertible and

\[ A^{-1} = E_1 (-3) E_2 (-2) E_3 \left( \frac{1}{3} \right) E_{32} (3) E_{31} (-4) E_{12} I_3 \]

which means that \( A^{-1} \) is obtained from \( I_3 \) by doing the same row operations that were performed to transform \( A \) into \( I_3 \).

Remark. The above argument can be performed in general and shows that the theorem is true, see the textbook.