Continuous Dependence on Parameters of the Fixed Points Set for some Set-Valued Operators

EDUARD KIRR and ADRIAN PETRUȘEL
“Babeș-Bolyai” University, Department of Mathematics, str. M. Kogălniceanu nr. 1, 3400 Cluj-Napoca, Romania
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Abstract. In this paper we extend the notion of $I^0$–continuity and uniform $I^0$–continuity from [2] to set-valued operators. Using these properties, we prove some results on continuous dependence of the fixed points set for families of contractive type set-valued operators.


Key Words: $I^0$–continuity, semi-continuous set-valued operators, Hausdorff-Pompeiu generalized semi-metric, contractive type set-valued operators.

1 Introduction

The notions of $I^0$–continuous and uniform $I^0$–continuous operators were first introduced by I. Del Prete and C. Esposito in [2] for single-valued operators on metric spaces. In order to study the fixed points set of a larger class of maps, in [4] the author extends the above notions for single-valued operators on uniform spaces. In the second section, our purpose is to introduce the $I^0$–continuity, uniform $I^0$–continuity and locally uniform $I^0$–continuity for set-valued operators. We point out that the notions and the results of this paper can be obtained in the frame of uniform spaces, but, for the sake
of simplicity, we will consider only the case of set-valued operators defined on metric spaces.

In the third section, we shall prove three proposition and a corollary concerning the continuous dependence on parameters of the fixed points set for abstract $I^0$ – continuous or locally uniform $I^0$ – continuous set-valued operators. These will be the basis for the main results of the paper.

The last section is dedicated to our main theorems. We shall deduce the continuous dependence of the fixed points set for families of $\varphi$ – contractions (generalized contractions) and for families of contractive set-valued operators. Relying on the fact that such operators are $I^0$ – continuous, the theorems will be a direct consequence of the abstract results from the third section.

Finally, we recall some notations and definitions. Let $(X, d)$ be a metric space, we denote:

$$ B(x, r) = \{x' \in X; d(x, x') < r\}, \quad \text{where } x \in X \text{ and } r > 0; $$

$$ 2^X = \{A; A \subseteq X\}; $$

$$ d(x, A) = \begin{cases} 
\inf\{d(x, a); a \in A\} & \text{if } x \in X \text{ and } A \in 2^X \backslash \{\emptyset\}, \\
+\infty & \text{if } x \in X \text{ and } A = \emptyset,
\end{cases} $$

and we consider the Hausdorff-Pompeiu generalized semi-metric:

$$ D: 2^X \times 2^X \to \mathbb{R}_+ \cup \{+\infty\} $$

$$ D(A, B) = \begin{cases} 
\max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\} & \text{if } A \neq \emptyset \neq B, \\
0 & \text{if } A = \emptyset = B, \\
+\infty & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B.
\end{cases} $$

Also, for a set-valued operator $F: X \to 2^X$, let $\text{Fix}F = \{x \in X; x \in F(x)\}$ be the set of all fixed points of $F$.

## 2 $I^0$ – continuous Operators

Let $d$ be a metric on the nonempty set $X$ and consider a set-valued operator $F: X \to 2^X$.

**Definition 2.1** We say that $F$ is $I^0$ – continuous if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ with the property that the following implication is true:

$$ \forall x \in X \text{ with } d(x, F(x)) < \delta \Rightarrow \exists x^* \in \text{Fix}F \text{ such that } d(x, x^*) < \varepsilon. $$
In particular, if $F$ is single-valued, continuous and it has at least one fixed point, then our definition is equivalent with that from [2, p.190].

Now, consider the set $Y$ of the parameters and the set-valued operator $F : X \times Y \to 2^X$, where the nonempty set $X$ is endowed with the metric $d$.

**Definition 2.2** We say that $F$ is uniform $I^0$–continuous if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that the following implication is true:

$$
\forall (x, y) \in X \times Y \text{ with } d(x, F(x, y)) < \delta \Rightarrow \exists x^* \in \text{Fix}F(\cdot, y) \text{ with } d(x, x^*) < \varepsilon.
$$

In the case that $F(\cdot, y) : X \to 2^X$ is single-valued, continuous and it has at least one fixed point, for every $y \in Y$, then one will get the definition from [2, p.191] after reformulating ours.

The restrictive conditions on uniform $I^0$–continuous operators can be weakened if we assume that the set of the parameters is a topological space.

**Definition 2.3** We say that $F$ is locally uniform $I^0$–continuous at $y_0 \in Y$ if and only if there is a neighborhood $V$ of $y_0$ such that, for every $\varepsilon > 0$, there exists $\delta > 0$ with the property that the following implication is true:

$$
\forall (x, y) \in X \times V \text{ with } d(x, F(x, y)) < \delta \Rightarrow \exists x^* \in \text{Fix}F(\cdot, y) \text{ with } d(x, x^*) < \varepsilon.
$$

In the last section we shall deal with some classes of set-valued operators which are $I^0$–continuous, uniform $I^0$–continuous or locally uniform $I^0$–continuous.

## 3 Abstract Results

Throughout this section, $d$ will be a metric on the nonempty set $X$ and $(Y, \tau)$ the topological space of the parameters. We consider the set-valued operator $F : X \times Y \to 2^X$ and we attach to it the multifunction $p : Y \to 2^X$ given by:

$$
p(y) = \text{Fix}F(\cdot, y), \text{ for every } y \in Y.
$$

In what follows, we shall study the semi-continuity properties of the multifunction $p$ relying on the locally uniform $I^0$–continuous or $I^0$–continuous properties of the set-valued operator $F$. The next results can be regarded as extensions for set-valued operators of the main results from [4], provided that we would formulate them in the more general frame of uniform spaces. We
have agreed not to do this, in order to make this section more accessible and because the applications from the next section deal with set-valued operators on metric spaces.

**Proposition 3.1** Let \( y_0 \in Y \) and assume that:

1. \( F(x, \cdot) : Y \to 2^X \) is lower semi-continuous at \( y_0 \), for every \( x \in X \);
2. \( F \) is locally uniform \( I^0 \)–continuous at \( y_0 \).

Then \( p : Y \to 2^X \) is lower semi-continuous at \( y_0 \).

**Proof.** We have to show that, for each \( x \in p(y_0) \) and for every neighborhood \( U(x_0) \) of \( x_0 \) in \( X \), there exists a neighborhood \( V \) of \( y_0 \) such that \( p(y_0) \cap U(x_0) \neq \emptyset \), for every \( y \in V \). In order to do this, let us choose arbitrarily \( x_0 \in p(y_0) \) and a neighborhood \( U(x_0) \) of \( x_0 \) in \( X \). Then, there exists \( \varepsilon > 0 \) with the property that \( B(x_0, \varepsilon) \subset U(x_0) \). Now, from (ii) and Definition 2.3, we obtain the neighborhood \( V' \) of \( y_0 \) and the number \( \delta > 0 \) for which the following implication is true:

\[
\forall (x, y) \in X \times V' \text{ with } d(x, F(x, y)) < \delta \Rightarrow \exists x^* \in p(y) \text{ with } d(x, x^*) < \varepsilon.
\]

Nevertheless, using (i), we can construct the neighborhood \( V'' \) of \( y_0 \) such that \( F(x_0, y) \cap B(x_0, \delta) \neq \emptyset \), for every \( y \in V'' \). Let \( V = V' \cap V'' \). Then, for each \( y \in V \), we have \( d(x_0, F(x_0, y)) < \delta \), so, there exists \( x^* \in p(y) \) with \( d(x_0, x^*) < \varepsilon \) or, equivalently, \( x^* \in p(y) \cap B(x_0, \varepsilon) \subset p(y) \cap U(x_0) \). Thus, \( p(y) \cap U(x_0) \neq \emptyset \), for every \( y \in V \).

Because \( x_0 \in p(y) \) and the neighborhood \( U(x_0) \) of \( x_0 \) in \( X \) were arbitrarily chosen, the proposition is completely proved.

**Remark 3.1.** The above result remains true if, instead of (i), we suppose that \( F(x, \cdot) : Y \to 2^X \) is continuous at \( y_0 \), for every \( x \in X \), where the topology on \( 2^X \) is given by the generalized semi-metric \( D \). Clearly, this condition implies (i).

In the next proposition we shall weaken the assumption (ii), but we will have to replace (i) with another condition.

**Proposition 3.2** Let \( y_0 \in Y \) and assume that:
(iii) \( F(x, \cdot): Y \to 2^X \) is upper semi-continuous at \( y_0 \), uniform with respect to \( x \in X \);

(iv) \( F(\cdot, y_0): X \to 2^X \) is \( I^0 \) - continuous.

Then, for each \( \varepsilon > 0 \), there exists a neighborhood \( V \) of \( y_0 \) such that:

\[
p(y) \subset \bigcup_{x \in p(y_0)} B(x, \varepsilon), \quad \text{for every } y \in V.
\]

Proof. Let \( \varepsilon > 0 \) be arbitrarily fixed. Clearly, (iv) implies the existence of \( \delta > 0 \) such that if \( x \in X \) and \( d(x, F(x, y_0)) < \delta \), then there is at least one \( x^* \in p(y_0) \) with \( d(x, x^*) < \varepsilon \). Now, using (iii), we can construct a neighborhood \( V \) of \( y_0 \) with the property that \( F(x, y) \subset \bigcup_{x' \in F(x, y_0)} B(x', \delta) \), for every \( (x, y) \in X \times V \). Consequently, for each \( (y, x) \in V \times p(y) \) we have \( d(x, F(x, y_0)) < \delta \). Then, taking into account the property of \( \delta \), for each \( (y, x) \in V \times p(y) \), there is \( x^* \in p(y_0) \) such that \( x \in B(x^*, \varepsilon) \) or, equivalently, \( p(y) \subset \bigcup_{x^* \in p(y_0)} B(x^*, \varepsilon) \), for every \( y \in V \).

Nevertheless, \( \varepsilon > 0 \) was arbitrarily fixed, so the proof is complete.

Remark 3.2. (a) As we can see from the proof, Proposition 3.2 remains true if we replace the assumption (iii) with the following weaker one:

(v) for each \( \delta > 0 \) there is a neighborhood \( V \) of \( y_0 \) such that \( F(x, y) \subset \bigcup_{x' \in F(x, y_0)} B(x', \delta) \), for every \( (x, y) \in X \times V \).

(b) The conclusion of the Proposition 3.2 still holds, even if, instead of (iii), we suppose that \( F(x, \cdot): Y \to 2^X \) is continuous at \( y_0 \), uniform with respect to \( x \in X \), where \( 2^X \) is endowed with the topology induced by the Hausdorff-Pompeiu generalized semi-metric. Indeed, this condition implies (v) and the rest is a direct consequence of (a).

In spite of its technical formulation, Proposition 3.2 is very useful in studying the semi-continuity of \( p \), as we shall see from the next result.

Corollary 3.3 Suppose that the hypothesis (iv), (v) are satisfied and the set \( p(y_0) \) is compact. Then \( p: Y \to 2^X \) is upper semi-continuous at \( y_0 \).
Proof. We must demonstrate that, for each open set $U \subseteq X$ with $U \supset p(y_0)$, there exists a neighborhood $V$ of $y_0$ such that $p(y) \subset U$, for every $y \in V$. Let us choose arbitrarily an open set $U$ in $X$ with $U \supset p(y_0)$. From the compactness of $p(y_0)$, we get the existence of $\varepsilon > 0$ with the property that: $\cup_{x \in p(y_0)} B(x, \varepsilon) \subset U$ (see [3, p.234, Theorem 4.5]). Now, using Proposition 3.2, we can construct a neighborhood $V$ of $y_0$ such that $p(y) \subset \cup_{x \in p(y_0)} B(x, \varepsilon)$, for every $y \in V$. Hence, $p(y) \subset U$, for every $y \in V$.

Due to the arbitrary choice of the open set $U$ in $X$ with $U \supset p(y_0)$, the proof is complete.

Nevertheless, if condition (ii) holds, $p(y_0)$ is compact and we assume that $F(x, \cdot) : Y \to 2^X$ is continuous as a multifunction (i.e. lower and upper semi-continuous) at $y_0$, uniform with respect to $x \in X$, then, using the above Corollary and Proposition 3.1, we deduce the continuity of $p : Y \to 2^X$ as a multifunction. But more interesting is the next result.

**Proposition 3.4** Consider on $2^X$ the topology induced by the Hausdorff-Pompeiu generalized semi-metric $D$. Assume that (ii) holds for an $y_0 \in Y$ and $F(x, \cdot) : (Y, \tau) \to (2^X, D)$ is continuous at $y_0$, uniform with respect to $x \in X$.

Then $p : (Y, \tau) \to (2^X, D)$ is continuous at $y_0$.

**Proof.** There are two cases.

In the first case, we assume that $p(y_0)$ is empty. We shall show that there exists a neighborhood $V$ of $y_0$ such that $p(y)$ is empty, for every $y \in V$. Clearly, (ii) implies (iv) and, using Remark 3.2.(b), we can choose the neighborhood $V$ of $y_0$ with the property that $p(y) \subset \cup_{x \in p(y_0)} B(x, 1) = \emptyset$, where the last equality follows from $p(y_0) = \emptyset$. Thus, $p(y) = \emptyset$, for every $y \in V$. Now, by the definition of $D$, we have $D(p(y), p(y_0)) = D(\emptyset, \emptyset) = 0$, $\forall y \in V$, hence, $p : (Y, \tau) \to (2^X, D)$ is continuous at $y_0$, in this case.

In the second case, we assume that $p(y_0)$ is nonempty. Let us fix $\varepsilon > 0$ arbitrarily. Because (ii) implies (iv), we may apply Remark 3.2.(b) and we deduce that there exists a neighborhood $V'$ of $y_0$ such that $p(y) \subset \cup_{x \in p(y_0)} B(x, \varepsilon)$, $\forall y \in V'$, or, equivalently, $\sup_{x \in p(y)} d(x, p(y_0)) \leq \varepsilon$, for every $y \in V'$. On the other hand, using (ii), Remark 3.1. and arguing as in the proof of Proposition 3.1, we can construct the neighborhood $V''$ of $y_0$ with the property that, for each $(y, x) \in V'' \times p(y_0)$, we have $p(y) \cap B(x, \varepsilon) \neq \emptyset$. Consequently, $d(x, p(y)) < \varepsilon$, $\forall (y, x) \in V'' \times p(y_0)$, hence, $\sup_{x \in p(y_0)} d(x, p(y)) \leq \varepsilon$, for every $y \in V''$. 

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Now, let $V = V' \cap V''$. Then, for each $y \in V$, we get:

$$D(p(y), p(y_0)) = \max \{ \sup_{x^* \in p(y)} d(x^*, p(y_0)), \sup_{x \in p(y_0)} d(x, p(y)) \} \leq \varepsilon.$$  

Due to the arbitrary choice of $\varepsilon > 0$, we conclude that $p : (Y, \tau) \to (2^X, D)$ is continuous at $y_0$ in this case, too.

We have covered all possible cases, so the proof is complete.

4 Main Theorems

In this section, we shall prove some extensions for generalized set-valued contractions of the classical dependence on parameters of the fixed point for contractions. Such extensions were obtained in [4] and [6, Chapter VII, pp.57-64] for the case of single-valued operators.

For convenience, let us recall the definitions of set-valued $\varphi$-contraction and of set-valued contractive operator:

Definition 4.1 Let $(X, d)$ be a metric space, $D$ the Hausdorff-Pompeiu generalized semi-metric on $2^X$ and $F : X \to 2^X$ a set-valued operator with nonempty and closed values. Then:

(a) $F$ is contractive if and only if

$$D(F(x_1), F(x_2)) < d(x_1, x_2), \quad \text{for every } x_1, x_2 \in X, \ x_1 \neq x_2.$$  

(b) $F$ is a $\varphi$-contraction if and only if there exists a strictly increasing mapping $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{k=1}^{\infty} \varphi^k(t) < \infty$, for every $t > 0$ and

$$D(F(x_1), F(x_2)) \leq \varphi(d(x_1, x_2)), \quad \text{for every } x_1, x_2 \in X,$$

where $\varphi^k = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_k$.

We need two auxiliary lemmas, which, in particular, show that there are nice examples of $I^0$ – continuous and uniform $I^0$ – continuous set-valued operators:
Lemma 4.2 Let \((X, d)\) be a complete metric space and \(F : X \to 2^X\) a set-valued operator with nonempty and closed values. If one of the following hypothesis holds:

(a) \(F\) is \(\varphi\)–contraction;

(b) \(X\) is compact and \(F\) is contractive;

then \(F\) is \(I^0\) – continuous.

Proof. (a) Let \(\varepsilon > 0\). Since \(\sum_{k=1}^{\infty} \varphi^k(1) < \infty\), there exists \(N(\varepsilon) \in \mathbb{N}^*\) such that \(\sum_{k=N(\varepsilon)}^{\infty} \varphi^k(1) < \varepsilon / 2\). We shall prove that \(F\) verifies Definition 2.1 with \(\delta = \varphi^{N(\varepsilon)}(1)\). Clearly, \(\delta > 0\), because, from \(0 < 1\), using the positivity and strict monotonicity of \(\varphi\), we have \(0 \leq \varphi^{N(\varepsilon)}(0) < \varphi^{N(\varepsilon)}(1) = \delta\). It remains to show that, for each \(x \in X\), condition \(d(x, F(x)) < \delta\) implies the existence of \(x^* \in \text{Fix}F\) with the property: \(d(x, x^*) < \varepsilon\).

In order to do this, we shall use some ideas from the proof of Wegrzyk’s Theorem (see [8, Theorem 2.1.] for more details). Let us fix arbitrarily \(x_0 \in X\) with \(d(x_0, F(x_0)) < \delta\). Then, there is \(x_1 \in F(x_0)\) such that \(d(x_0, x_1) < \delta\).

Now, we can choose \(x_2 \in F(x_1)\), which verifies:

\[
d(x_1, x_2) < D(F(x_0), F(x_1)) + \varphi(\delta) - \varphi(d(x_0, x_1)) \leq \varphi(d(x_0, x_1)) + \varphi(\delta) - \varphi(d(x_0, x_1)) = \varphi(\delta),
\]

where we have used that \(F\) is \(\varphi\)–contraction, \(\varphi\) is strictly increasing and Lemma 1.3. from [8]. In the same way, we can construct inductively the sequence \((x_n)_{n \in \mathbb{N}} \subset X\) such that:

1. \(d(x_n, x_{n+1}) < \varphi^n(\delta)\), for every \(n \in \mathbb{N}\), where \(\varphi^n(\delta) := \delta^n\);

2. \(x_{n+1} \in F(x_n)\), for every \(n \in \mathbb{N}\).

Now, from \(\sum_{k=1}^{\infty} \varphi^k(\delta) < \infty\) and 1., we deduce that the sequence \((x_n)_{n \in \mathbb{N}}\) converges to a point \(x^* \in X\), which, by 2., the continuity of \(F : (X, d) \to (2^X, D)\) and the fact that \(F\) has closed values, is a fixed point of \(F\). Moreover, for each \(n \in \mathbb{N}^*\), using 1., we have successively:

\[
d(x_0, x^*) \leq d(x_0, x_n) + d(x_n, x^*) \leq \sum_{k=0}^{n-1} d(x_k, x_{k+1}) + d(x_n, x^*) < \sum_{k=0}^{n-1} \varphi^k(\delta) + d(x_n, x^*) = \sum_{k=0}^{n-1} \varphi^k(\varphi^{N(\varepsilon)}(1)) + d(x_n, x^*) < \sum_{k=N(\varepsilon)}^{\infty} \varphi^k(1) + d(x_n, x^*) < \varepsilon / 2 + d(x_n, x^*).
\]
Hence, \(d(x_0, x^*) < \varepsilon/2 + d(x_n, x^*)\) and, tacking the limit when \(n \to \infty\), we obtain \(d(x_0, x^*) \leq \varepsilon/2 < \varepsilon\).

Due to the arbitrary choice of \(\varepsilon > 0\) and \(x_0 \in X\), we have proved that \(F\) verifies Definition 2.1. So, \(F\) is \(I^0 -\) continuous.

(b) Suppose contrary, there exists \(\varepsilon > 0\) such that, for each \(\delta > 0\), there is \(x_\delta \in X\) with the properties: \(d(x_\delta, F(x_\delta)) < \delta\) and \(B(x_\delta, \varepsilon) \cap \text{Fix} F = \emptyset\).

Let us consider the sequence \((x_n)_{n \in \mathbb{N}}\) such that \(d(x_n, F(x_n)) \leq 1/n\) and \(B(x_n, \varepsilon) \cap \text{Fix} F = \emptyset\). \(X\) being a compact set, there is a subsequence \((x_{n_k})_{k \in \mathbb{N}} \subseteq (x_n)_{n \in \mathbb{N}}\) convergent to a point \(x^* \in X\).

Now, from \(F\) contractive with closed values and \(X\) compact, one obtain that \(F\) is upper semi-continuous (see [7]), hence, by Proposition 2.1 from [5, p.29], the mapping \(h : X \to \mathbb{R}_+\), \(h(x) = d(x, F(x))\) is lower semi-continuous. Consequently, \(h(x^*) \leq \liminf_{k \to \infty} h(x_{n_k}) = 0\), which implies \(h(x^*) = 0\) or, equivalently, \(x^* \in \text{Fix} F\).

Since \(\lim_{k \to \infty} x_{n_k} = x^*\), we have that there is \(k_0 \in \mathbb{N}\) such that \(d(x_{n_k}, x^*) < \varepsilon\), for every \(k \geq k_0\), which contradicts the assumption \(B(x_n, \varepsilon) \cap \text{Fix} F = \emptyset\), for every \(k \in \mathbb{N}\).

**Lemma 4.3** Let \((X, d)\) be a complete metric space, \((Y, \tau)\) be the topological space of the parameters and \(F : X \times Y \to 2^X\) a set-valued operator such that \(F(x, y)\) is nonempty and closed, for every \((x, y) \in X \times Y\). If \(F(\cdot, y) : X \to 2^X\) is a \(\varphi-\) contraction with \(\varphi\) independent on \(y\), then \(F\) is uniform \(I^0 -\) continuous.

**Proof.** From the proof of Lemma 4.2, we know that, for each \(y \in Y\), \(F(\cdot, y)\) verifies Definition 2.1 with \(\delta = \varphi^{N(\varepsilon)}(1)\), where \(N(\varepsilon)\) satisfies: \(\sum_{k=N(\varepsilon)}^{\infty} \varphi^k(1) < \varepsilon/2\). Since \(\varphi\) does not depend on \(y\), we deduce that \(\delta\) does not depend on \(y\). So, \(F\) satisfies Definition 2.2, which means that \(F\) is uniform \(I^0 -\) continuous.

Now, we are able to prove our main results:

**Theorem 4.4** Let \((X, d)\) be a complete metric space, \(D\) the Hausdorff-Pompeiu generalized semi-metric on \(2^X\), \((Y, \tau)\) the topological space of the parameters, \(y_0 \in Y\) and \(F : X \times Y \to 2^X\) a set-valued operator with nonempty and closed values. Consider \(p : Y \to 2^X\) given by \(p(y) = \text{Fix} F(\cdot, y)\).

(a) If \(F(x, \cdot) : (Y, \tau) \to (2^X, D)\) is continuous at \(y_0\), uniform with respect to \(x \in X\), and \(F(\cdot, y_0) : X \to 2^X\) is a \(\varphi-\) contraction, then, for each \(\varepsilon > 0\),
there exist a neighborhood $V$ of $y_0$ such that $p(y) \subseteq \bigcup_{x \in p(y_0)} B(x, \varepsilon)$, for every $y \in V$. Moreover, $p : Y \to 2^X$ is upper semi-continuous at $y_0$, provided that $p(y_0)$ is compact.

(b) If $F(x, \cdot) : (Y, \tau) \to (2^X, D)$ is continuous at $y_0$, for every $x \in X$, and, for each $y \in Y$, $F(\cdot, y) : X \to 2^X$ is a $\varphi$--contraction, where $\varphi$ is independent on $y$, then $p : Y \to 2^X$ is lower semi-continuous at $y_0$.

(c) If $F(x, \cdot) : (Y, \tau) \to (2^X, D)$ is continuous at $y_0$, uniform with respect to $x \in X$, and, for each $y \in Y$, $F(\cdot, y) : X \to 2^X$ is a $\varphi$--contraction, where $\varphi$ is independent on $y$, then $p : (Y, \tau) \to (2^X, D)$ is continuous at $y_0$.

Proof. (a) From Lemma 4.2(a), we know that $F(\cdot, y_0)$ is $I^0$--continuous. The conclusion follows immediately by Remark 3.2.(b) and Corollary 3.3.

(b) Lemma 4.3 implies that $F$ is uniform $I^0$--continuous. Now, the conclusion is a direct consequence of Remark 3.1.

(c) Using Lemma 4.3, we deduce that the hypothesis of Proposition 3.4 are satisfied. So, the proof is complete.

Remark 4.4.(a) The result of Theorem 4.4(a) remains true if we replace the continuity assumption on the second argument of $F$ with $F(x, \cdot) : Y \to 2^X$ is upper semi-continuous at $y_0$, uniform with respect to $x \in X$, or, more generally, for each $\varepsilon > 0$, there exists a neighborhood $V(\varepsilon)$ of $y_0$, which depends only on $\varepsilon$, such that:

$$F(x, y) \subseteq \bigcup_{x' \in F(x, y_0)} B(x', \varepsilon), \text{ for every } (x, y) \in X \times V(\varepsilon).$$

The proofs are analogous, using Proposition 3.2 or Remark 3.2.(a), respectively, instead of Remark 3.2.(b).

(b) Also, the result of Theorem 4.4(b) is still valid if we replace the continuity assumption on the second argument of $F$ with the weaker one: $F(x, \cdot) : Y \to 2^X$ is lower semi-continuous at $y_0$. The only change in the proof is the use of Proposition 3.1 instead of Remark 3.1.

(c) If $F$ is single-valued, then clearly, $p$ become single-valued and the conclusions of Theorem 4.4(a),(b) will be that $p : Y \to X$ is continuous at $y_0$. So, we have just reobtained the results from [4, Theorem 4.2.(b) and Theorem 4.4.(b)], but we have required stronger properties for $\varphi$ (see Definition 4.1).
Remark. One can adapt the techniques used in Lemmas 4.2, 4.3 and Theorem 4.4 in order to obtain analogous results for other types of generalized set-valued contractions, as, for example, in the case of locally contractive set-valued mapping from [1].

Theorem 4.5 Let \((X, d)\) be a compact metric space, \(D\) the Hausdorff-Pompeiu generalized semi-metric on \(2^X\), \((Y, \tau)\) the topological space of the parameters, \(y_0 \in Y\) and \(F : X \times Y \to 2^X\) a set-valued operator with nonempty and closed values. Consider \(p : Y \to 2^X\) given by \(p(y) = \text{Fix}F(\cdot, y)\).

(a) If \(y_0 \in Y\), \(F(x, \cdot) : (Y, \tau) \to (2^X, D)\) is continuous at \(y_0\), uniform with respect to \(x \in X\), and \(F(\cdot, y) : X \to 2^X\) is a contractive operator, then \(p : Y \to 2^X\) is upper semi-continuous at \(y_0\).

(b) If \(F(x, \cdot) : (Y, \tau) \to (2^X, D)\) is continuous at \(y_0\), for every \(x \in X\) and \(F(\cdot, y) : X \to 2^X\) is contractive, for every \(y \in Y\), then \(p\) is upper semi-continuous.

Proof. (a) By Lemma 4.2, we have that \(F(\cdot, y_0)\) is \(I^0\)-continuous. From \(F(\cdot, y_0) : (X, d) \to (2^X, D)\) continuous with closed values and \(X\) compact, one can easily obtain that \(p(y_0)\) is compact. So, the conclusion follows from Remark 3.2.(b) and Corollary 3.3.

(b) We shall show that \(F(x, \cdot) : (Y, \tau) \to (2^X, D)\) is continuous at each \(y \in Y\), uniform with respect to \(x \in X\). Then, applying the previous result, we deduce that \(p : Y \to 2^X\) is upper semi-continuous at each \(y \in Y\), which means that \(p\) is upper semi-continuous.

Let us fix arbitrarily \(\varepsilon > 0\). We have to construct for each \(y \in Y\) a neighborhood \(V(y)\) of \(y\), which depends only on \(y\), such that \(D(F(x, y'), F(x, y)) < \varepsilon\), for every \((x, y') \in X \times V(y)\). In order to do this, let us choose arbitrarily \(y_0 \in Y\). Then, for each \(x \in X\), by the continuity of \(F(x, \cdot)\), there exists a neighborhood \(V(x)\) of \(y_0\) such that \(D(F(x, y), F(x, y_0)) < \varepsilon/3\), for every \(y \in V(x)\). Using the contractive assumption, we get:

\[
D(F(x', y), F(x', y_0)) \leq D(F(x', y), F(x, y)) + D(F(x, y), F(x, y_0)) + D(F(x, y_0), F(x', y_0)) < d(x', x) + \varepsilon/3 + d(x', x)
< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,
\]

for every \((x', y) \in B(x, \varepsilon/3) \times V(x)\).

Clearly, \(\bigcup_{x \in X} B(x, \varepsilon/3)\) is an open cover of the compact space \(X\). Hence, there exist \(n \in \mathbb{N}^*\) and \(x_1, x_2, \ldots, x_n \in X\) such that \(X = \bigcup_{j=1}^n B(x_j, \varepsilon/3)\).
We denote $V(y_0) = \cap_{j=1}^n V(x_j)$, which is also a neighborhood of $y_0$. Now, for each $(x', y) \in X \times V(y_0)$, there is $j \in \{1, 2, \ldots, n\}$ such that $(x', y) \in B(x_j, \varepsilon/3) \times V(x_j)$. So, by the above inequalities, with $x := x_j$, we have $D(F(x, y), F(x, y_0)) < \varepsilon$.

Due to the arbitrary choice of $\varepsilon > 0$ and $y_0 \in Y$, we have just proved that $F(x, \cdot): (Y, \tau) \rightarrow 2^X$ is continuous at each $y \in Y$, uniform with respect to $x \in X$. The conclusion follows as we showed at the beginning.

Remark 4.5.(a) We may replace the uniform continuity of family $\{F(x, \cdot)\}_{x \in X}$ at $y_0$ with respect to the topology induced by the generalized semi-metric $D$ on $2^X$, with the weaker assumption: $F(x, \cdot): Y \rightarrow 2^X$ is upper semi-continuous at $y_0$, uniform with respect to $x \in X$, and the conclusion of Theorem 4.5(a) is still true.

(b) Also, in Theorem 4.5(b), the continuity of $F(x, \cdot): (Y, \tau) \rightarrow (2^X, D)$ can be weakened at: $F(x, \cdot): Y \rightarrow 2^X$ is upper semi-continuous, and the result still holds.

(c) Theorem 4.5 is a generalization of Theorem 4.3.(d) from [4] and a solution of the open problem 7.3.2. in [6, p.63] for set-valued operators. Indeed, if $F$ is single-valued, then $p: Y \rightarrow 2^X$ is single-valued and its upper semi-continuity implies its continuity.

References


