Periodic Solutions for Perturbed Hamiltonian Systems with Superlinear Growth and Impulsive Effects

Eduard Kirr
“Babeș-Bolyai” University, Department of Mathematics, 1 M. Kogălniceanu, RO-3400 Cluj-Napoca, Romania.
e-mail ekirr@math.ubbcluj.ro

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Abstract
The aim of this paper is to prove the existence of periodic piecewise continuous solutions for impulsive planar differential systems having the form of a perturbed Hamiltonian. The proof relies on a continuation technique introduced in [1] and adapted for impulsive equations in [2].

1 Introduction
In this paper we shall prove the existence of at least one piecewise continuous solution for the periodic boundary value problem:

\[
\begin{cases}
    x'(t) = J\nabla V(x(t)) + q(t, x(t)) \quad \text{for a.e. } t \in [0, 1], \\
    x(t_k^+) = \psi^k(x(t_k)) \quad \text{for } k \in \mathbb{1, m}, \\
    x(0) = x(1),
\end{cases}
\]

where \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), \( 0 < t_1 < t_2 < \ldots < t_m < 1 \), are fixed points, in which the solutions are subject to impulsive effects, and we have denoted \( x(t^+) := \lim_{s \downarrow t} x(s) \). In what follows \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) will be the euclidean
scalar product, respective the euclidean norm in $\mathbb{R}^2$, and we shall suppose that:

(h1) $V : \mathbb{R}^2 \to \mathbb{R}$ is of class $C^1$ with the properties:

$$\lim_{\|z\|\to \infty} |\langle \nabla V(z), z \rangle | / \| z \|^2 = +\infty,$$

(2) and

$$\| \nabla V(z) \| \leq A \| V(z) \| + B, \text{ for all } z \in \mathbb{R}^2,$$

(3) for some fixed $A, B \in \mathbb{R}_+$;

(h2) $q : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}^2$ is a $L^1$–Carathéodory function, i.e. $q$ is Carathéodory and

$$\| q(t, z) \| \leq Q(t) \text{ for a.e. } t \in [0, 1] \text{ and all } z \in \mathbb{R}^2,$$

for a fixed function $Q \in L^1(0, 1; \mathbb{R}_+)$;

(h3) $\psi^k : \mathbb{R}^2 \to \mathbb{R}^2$ are continuous for every $k \in \overline{1, m}$ and there exist $n \in \mathbb{N}^*, \ r > 0$ such that

$$n \sum_{k=1}^{m} [\arg z_k - \arg \psi^k(z_k)] \equiv 1 \mod(2)$$

for all $(z_k)_{k=1}^m \in (\mathbb{R}^2)^m$ with $\| z_k \| \geq r$ and $\| \psi^k(z_k) \| \geq r, \ k \in \overline{1, m}$.

We point out that the superlinear character of (1) is given by condition (h1). Such problem have been already studied in [1], where no impulses were considered, and in [2, Example 3].Our main goal is to improve the result from [2, Example 3] using more deeply the properties of the planar Hamiltonian systems, namely the uniform rotation around the origin of the solutions. Those properties will allow us to relax the assumption on impulses from [2, Example 3] using (h3) instead. In order to obtain the existence of at least one piecewise continuous solution for (1) we shall need an abstract result proved in [4] in the frame of the topological transversality theory. For convenience we shall state this theorem.

Let $X$ be a real Banach space, $K \subseteq X$ a convex set and $H : K \times [0, 1] \to K$ a completely continuous map. Denote

$$S = \{(x, \lambda) \in K \times [0, 1]; \ H(x, \lambda) = x\}$$
and for any fixed \( x_0 \in K \), let

\[
S(x_0) = \{ x \in K ; (1 - \mu)x_0 + \mu H(x, 0) = x, \text{ for some } \mu \in [0, 1] \}.
\]

Also consider a continuous functional \( \Phi : K \times [0, 1] \rightarrow \mathbb{R} \). Then, we have the following theorem [4, Corollary 2]:

**Theorem 1** Assume

(i1’) \( \Phi \) is proper on \( S \);

(i2’) \( \Phi \) is bounded bellow on \( S \) and there is a sequence \((c_j)_{j \in \mathbb{N}}\) of real numbers such that \( c_j \rightarrow \infty \) and \( c_j \notin \Phi(S) \) for all \( j \in \mathbb{N} \);

(i3’) there is \( x_0 \in K \) such that \( S(x_0) \) is bounded.

Then, for each \( \lambda \in [0, 1] \), there exists at least one fixed point of \( H(\cdot, \lambda) \) in \( K \).

### 2 Main result

In this section we shall prove the existence of solutions for (1) in the following space of functions

\[
C_T = \{ x : [0, 1] \rightarrow \mathbb{R}^2 ; x \text{ is everywhere continuous except, eventually, the points } t_1, t_2, \ldots, t_m \text{ of discontinuity of first type at which } x \text{ is left continuous} \}.
\]

**Theorem 2** If (h1)-(h3) are satisfied, then the periodic boundary value problem (1) has at least one solution in \( C_T \).

**Proof:** Let us consider the family of periodic boundary value problems:

\[
\begin{align*}
    x'(t) &= J \nabla V(x(t)) + \lambda q(t, x(t)) \quad \text{for a.e. } t \in [0, 1], \\
    x(t_{k+}) &= \lambda \psi^k(x(t_k)) \quad \text{for } k \in \overline{1, m}, \\
    x(0) &= x(1).
\end{align*}
\]

In order to apply the Theorem 1 we choose \( X := C_T \) endowed with the usual \( C \)-norm, \( \| x \|_C = \sup\{ \| x(t) \|_t \in [0, 1] \} \). Notice that \( C_T \) can be identified with the real Banach space \( \prod_{k=0}^{m} C[t_k, t_{k+1}] \), where \( t_0 := 0 \) and \( t_{m+1} := 1 \).
Thus, $C_T$ is a Banach space too. Also, we choose $K := \{ x \in C_T; \ x(0) = x(1) \}$ the convex subset of $C_T$.

To construct the completely continuous map $H : K \times [0, 1] \to K$ we define

$$W_{p}^{1,1} = \{ x \in K; \ x \text{ is absolutely continuous on each } ]t_k, t_{k+1}[, \ k = 0, 1, \ldots, m \}.$$ 

and the linear map

$$L : W_{p}^{1,1} \to L^1(0, 1; \mathbb{R}^2) \times (\mathbb{R}^2)^m,$$

$$L(x) = (x', \{ x(t_k^+) \}_{1 \leq k \leq m}).$$

This map is invertible and to get its inverse

$$L^{-1} : L^1 \times (\mathbb{R}^2)^m \to K$$

we have to solve $m + 1$ initial value problems:

$$\begin{cases}
    x'(t) = y(t) & \text{for a.e. } t \in [t_k, t_{k+1}], \\
    x(t_k) = u_k,
\end{cases}$$

for $1 \leq k \leq m$, (recall that $t_{m+1} := 1$) and

$$\begin{cases}
    x'(t) = y(t) & \text{for a.e. } t \in [0, t_1], \\
    x(0) = x(1),
\end{cases}$$

where $y \in L^1$ and $u = \{u_k\}_{1 \leq k \leq m} \in (\mathbb{R}^2)^m$. Thus, the unique solution $x \in K$ to $L(x) = (y, u)$ is the function:

$$\begin{align*}
    x(t) &= u_k + \int_{t_k}^{t} y(s)ds & \text{for } t_k < t \leq t_{k+1}, \ 1 \leq k \leq m, \\
    x(t) &= x(1) + \int_{0}^{t} y(s)ds & \text{for } 0 \leq t \leq t_1.
\end{align*}$$

We also define the nonlinear map

$$N : K \times [0, 1] \to L^1(0, 1; \mathbb{R}^2) \times (\mathbb{R}^2)^m,$$

$$N(x, \lambda) = (J\nabla V(x) + \lambda q(\cdot, x); \lambda \psi^k(x(t_k))).$$

Then, under assumptions (h1) and (h2), $N$ is well-defined, continuous and bounded. Moreover, by (5) and Ascoli-Arzela’s theorem, the map $L^{-1}N : K \times [0, 1] \to K$ is completely continuous and we can choose

$$H = L^{-1}N.$$
Finally, we consider $\Phi : K \times [0,1] \to \mathbb{R}_+$ given by
\[
\Phi(x, \lambda) = \frac{1}{2\pi} \left| \int_0^1 \frac{(Jx(t), Jx(t) + \lambda q(t))}{\max\{r, \|x(t)\|^2\}} dt \right|
\]
where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the euclidean scalar product, respective the euclidean norm in $\mathbb{R}^2$. This functional is a modification of the classical map which counts the number of rotations around the origin of the continuous integral curves of a planar system (see [3]) and, clearly, $\Phi$ is continuous on $K \times [0,1]$.

Now, if we can prove that the hypothesis (i1')-(i3') are satisfied, then, applying theorem 1, we will deduce that $H(\cdot, 1)$ has at least one fixed point in $K$, which is equivalent with the existence of at least one solution in $C_T$ for the periodic boundary value problem (1).

Check of (i1'): Due to the completely continuity of $H$, $\Phi$ is proper on $S = \{(x, \lambda) \in K \times [0,1]; H(x, \lambda) = x\}$ if, for every $j \in \mathbb{N}$, $\Phi^{-1}([0,j]) \cap S$ is bounded. Indeed, let us consider an arbitrary compact set $P \subset \mathbb{R}_+$, then $P$ is bounded, so, there exists $j \in \mathbb{N}$ such that $P \subset [0,j]$. Hence, $\Phi^{-1}(P) \cap S \subset \Phi^{-1}([0,j]) \cap S$ and $\Phi^{-1}(P) \cap S$ is bounded as a subset of the bounded set $\Phi^{-1}([0,j]) \cap S$. Since $H$ is completely continuous, we have that $H(\Phi^{-1}(P) \cap S)$ is relatively compact. Moreover, from the definition of $S$, we deduce $\Phi^{-1}(P) \cap S \subset H(\Phi^{-1}(P) \cap S) \times [0,1]$ and, by Tychonov’s theorem, $\Phi^{-1}(P) \cap S$ is relatively compact. Now, $\Phi^{-1}(P)$ and $S$ are closed, because of the continuity of the maps $\Phi$ respectively $H$. Consequently, $\Phi^{-1}(P) \cap S$ is compact. Since $P$ was an arbitrary compact subset of $\mathbb{R}_+$, we have just proved that the restriction of $\Phi$ on $S$ is proper provided that $\Phi^{-1}([0,j]) \cap S$ is bounded for all $j \in \mathbb{N}$. So, it remains to verify that $\Phi^{-1}([0,j]) \cap S$ is bounded for all $j \in \mathbb{N}$ and we shall do this in two steps.

In the first step we shall show that for each $j \in \mathbb{N}$ there exists $r_j > 0$ such that, for every $(x, \lambda) \in S$, $\Phi(x, \lambda) \leq j$ implies $\inf\{\|x(t)\|; t \in [0,1]\} \leq r_j$, while, in the second step, we shall show that for each $r_j > 0$ there is $R_j \geq r_j$ such that, for every $(x, \lambda) \in S$, $\inf\{\|x(t)\|; t \in [0,1]\} \leq r_j$ implies $\sup\{\|x(t)\|; t \in [0,1]\} \leq R_j$. Clearly, from these two steps, we shall have that $\Phi^{-1}([0,j]) \cap S$ is bounded for all $j \in \mathbb{N}$, and the check of (i1') will be complete.

For the first step, we fix $j \in \mathbb{N}$ arbitrarily, and we choose
\[
r_j = \max\{r, (q+1)/2\pi, r_j'\}
\]
where
\[ q = \int_0^1 Q(t)dt \]
is the L\(^1\)-norm of \(Q\) and \(r_j'\) is such that:
\[ |\langle \nabla V(x), x \rangle| / \|x\| \geq 2\pi(j + 1) \quad (6) \]
for all \(x \in \mathbb{R}^2\) with \(\|x\| > r_j'\). Note that the existence of \(r_j'\) with the above property is a consequence of the superlinear growth of \(V\) (see the relation (2) from (h1)).

In order to prove that for every \((x, \lambda) \in S\), \(\Phi(x, \lambda) = j\) implies \(\inf\{\|x(t)\|; t \in [0, 1]\} \leq r_j\), let us suppose contrary, i.e. there is \((x, \lambda) \in S\), such that \(\Phi(x, \lambda) = j\) and \(\inf\{\|x(t)\|; t \in [0, 1]\} > r_j\). Then, by the definition of the functional \(\Phi\), we have successively:
\[
\Phi(x, \lambda) = \frac{1}{2\pi} \left| \int_0^1 \frac{\langle Jx(t), J\nabla V(x(t)) + \lambda q(t, x(t)) \rangle}{\max\{r, \|x(t)\|^2\}} dt \right| =
\[
\geq \frac{1}{2\pi} \left| \int_0^1 \frac{\langle x(t), \nabla V(x(t)) \rangle}{\|x(t)\|^2} + \lambda \frac{\langle x(t), q(t, x(t)) \rangle}{\|x(t)\|^2} dt \right| 
\[
\geq \frac{1}{2\pi} \left| \int_0^1 \frac{\langle x(t), \nabla V(x(t)) \rangle}{\|x(t)\|^2} dt \right| = \frac{1}{2\pi} \left| \int_0^1 \frac{\langle x(t), q(t, x(t)) \rangle}{\|x(t)\|^2} dt \right|.
\]
Since \(\|x(t)\| > r_j \geq r_j'\), for all \(t \in [0, 1]\), we can apply, in the first term of the above relation, the inequality (6). Using also the Cauchy-Schwarz inequality for the scalar product in the second term, we get:
\[
\Phi(x, \lambda) \geq j + 1 - \lambda \int_0^1 \frac{\|q(t, x(t))\|}{2\pi \|x(t)\|^2} dt
\]
Now, from \(2\pi \|x(t)\| > 2\pi r_j \geq q + 1\), for all \(t \in [0, 1]\), and (h2) using also the fact that \(q\) is the L\(^1\)-norm of \(Q\), we obtain:
\[
\Phi(x, \lambda) \geq j + 1 - \lambda \int_0^1 \frac{Q(t)}{q + 1} dt = j + 1 - \lambda \frac{q}{q + 1} > j
\]
which contradicts \(\Phi(x, \lambda) = j\).

For the second step we shall prove the more general result:

**Lemma 3** If (h1)-(h3) are satisfied, then, for each \(\bar{r} \geq 0\), there exists \(\bar{R} \geq \bar{r}\) such that, for every \((x, \lambda) \in S\) with \(\inf\{\|x(t)\|; t \in [0, 1]\} \leq \bar{r}\), we have \(\sup\{\|x(t)\|; t \in [0, 1]\} \leq \bar{R}\).
Proof: First of all, using the Cauchy-Schwarz inequality and relation (2), we deduce that $$\lim_{\|z\| \to \infty} \| \nabla V(z) \| = \infty$$, hence, by (3), we have $$\lim_{\|z\| \to \infty} |V(z)| = \infty$$. So, there exists $$r' \geq 0$$ such that $$|V(z)| > 0$$ whenever $$\|z\| \geq r'$$.

Now, let us fix $$\tilde{r} \geq 0$$, and $$(x, \lambda) \in S$$ such that $$\inf\{\|x(t)\|; t \in [0,1]\} \leq \tilde{r}$$. Since $$x \in C_T$$, the restriction of $$x$$ on each interval $$[t_k, t_{k+1}]$$, $$0 \leq k \leq m$$, can be prolonged to a continuous function $$x_k$$ on $$[t_k, t_{k+1}]$$. Consequently, there are $$k \in [0, m]$$, and $$s \in [t_k, t_{k+1}]$$, with the property that $$|x_k(s)| = \inf\{\|x(t)\|; t \in [0,1]\}$$.

In what follows we shall construct an upper bound for the set $$\{\|x(t)\|; t \in [t_k, t_{k+1}]\}$$. We have only two possibilities: either $$\|x(t)\| \leq r'$$, for all $$t \in [t_k, t_{k+1}]$$, or there is an $$t \in [t_k, t_{k+1}]$$ such that $$\|x(t)\| > r'$$. In the second case we can find $$s_0, s_1 \in [t_k, t_{k+1}]$$ with the properties

$$\|x_k(s_0)\| = \max\{\tilde{r}, r'\},$$

$$\|x_k(s_1)\| = \sup\{\|x(t)\|; t \in [t_k, t_{k+1}]\},$$

$$\|x_k(t)\| \geq r'$$ for all $$t \in [s_0, s_1]$$ (or $$s_1, s_0$$).

If we denote

$$v(t) = \ln |V(x_k(t))| \quad \text{for all } t \in [s_0, s_1] \text{ (or } s_1, s_0\text{)},$$

we can write

$$v(s_1) = v(s_0) + \int_{0}^{1} v'(t)dt \leq v(s_0) + \text{sgn}(s_1 - s_0) \int_{0}^{1} |v'(t)|dt. \quad (7)$$

But, using the fact that $$(x, \lambda)$$ verifies (2), we have for every $$t \in [s_0, s_1]$$ (or $$s_1, s_0$$):

$$|v'(t)| = \left| \frac{\langle \nabla V(x(t)), x'(t) \rangle}{V(x(t))} \right| = \left| \frac{\langle \nabla V(x(t)), J\nabla V(x(t)) + \lambda q(t, x(t)) \rangle}{V(x(t))} \right| = \lambda \left| \frac{\langle \nabla V(x(t)), q(t, x(t)) \rangle}{V(x(t))} \right|.$$
Hence, by the Cauchy-Schwarz inequality, relation (3) and assumption (h2), we obtain

\[ |v'(t)| \leq \lambda |q(t, x(t))| \frac{\nabla V(x(t))}{|V(x(t))|} \leq \lambda Q(t)(A + B). \]

Replacing the last inequality in (7), we get

\[ v(s_1) \leq v(s_0) + q(A + B). \]

Now, the properness of the function \( \ln |V| \) (recall that \( \lim_{\|z\| \to \infty} |V(z)| = \infty \)) implies that there exists an \( R_1 \geq 0 \), depending only on \( v(s_0) + \lambda q(A + B) \), such that \( \|x_k(s_1)\| = \sup \{\|x(t)\| : t \in [t_k, t_{k+1}]\} \leq R_1 \).

Anyway, choosing \( \tilde{R}_1 = \max \{r', R_1\} \), we have

\[ \sup \{\|x(t)\| : t \in [t_k, t_{k+1}]\} \leq \tilde{R}_1 \]

where \( \tilde{R}_1 \) depends only on \( r \) (and the fixed constants \( r', q, A \) and \( B \)) regardless the two possible cases. Consequently, \( \|x(t_{k+1})\| \leq \tilde{R}_1 \) which implies \( \|x(t_{k+1})\| = \|x(0)\| \leq \tilde{R}_1 \), if \( k = m \) (recall that \( t_{m+1} := 1 \)) or, using the continuity of the functions \( \psi_k \), \( 1 \leq k \leq m \), \( \|x(t_{k+1}^k)\| \leq \sup_{k \in \mathbb{N}} \sup \{\|\psi_k(z)\| : \|z\| \leq \tilde{R}_1\} \).

Anyway,

\[ \inf \{\|x(t)\| : t \in [t_{k+1}, t_{k+2}]\} \leq \tilde{r}_1 \]

where \( \tilde{r}_1 = \max \left\{ \tilde{R}_1, \sup_{k \in \mathbb{N}} \sup \{\|\psi_k(z)\| : \|z\| \leq \tilde{R}_1\} \right\} \) depends only on \( \tilde{r} \) and if \( k = m \), then \( [t_{m+1}, t_{m+2}] \) must be interpreted as the interval \([0, 1]\).

The same arguments on \( [t_{k+1}, t_{k+2}] \) as on \( [t_k, t_{k+1}] \), where \( \tilde{r} \) will be replaced by \( \tilde{r}_1 \), will allow us to construct the positive numbers \( \tilde{R}_2 \) and \( \tilde{r}_2 \) such that

\[ \sup \{\|x(t)\| : t \in [t_{k+1}, t_{k+2}]\} \leq \tilde{R}_2 \]

and

\[ \inf \{\|x(t)\| : t \in [t_{k+2}, t_{k+3}]\} \leq \tilde{r}_2 \]

where \( \tilde{R}_2 \) and \( \tilde{r}_2 \) depends only on \( \tilde{r} \) by the intermediate number \( \tilde{r}_1 \), and if \( k = m \), then \( [t_{m+1}, t_{m+2}] \) must be interpreted as the interval \([0, 1]\). Continuing in the same way we can construct the finite sequence \( \tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_{m+1} \) which depends only on \( \tilde{r} \), and clearly \( \tilde{R} = \max \{\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_{m+1}\} \) will have the property

\[ \sup \{\|x(t)\| : t \in [0, 1]\} \leq \tilde{R}. \]
and will depend only on $\tilde{r}$.

Thus, the lemma is completely proved. $\square$

The lemma covers also the purpose of the second step, so, as we saw at the beginning, the check of (i1') is now complete.

Check of (i2'): Clearly, $\Phi$ is bounded below by 0, since $\Phi$ takes values only in $\mathbb{R}_+$. By the Lemma 3, there is a positive number $R$ such that $\sup\{\|x(t)\| \ ; \ t \in [0,1]\} \leq R$ whenever $(x,\lambda) \in S$, for some $\lambda \in [0,1]$, and $\inf\{\|x(t)\| \ ; \ t \in [0,1]\} \leq r$. Then, the properness of $\Phi$ on $S$ implies that the set $\Phi((x,\lambda) \in S; \|x(t)\| \leq R \text{ for all } t \in [0,1]))$ is bounded in $\mathbb{R}$. Let $j_0 \in \mathbb{N}$ be such that

$$(j_0 + 1/2)/n > \Phi((x,\lambda) \in S; \|x(t)\| \leq R \text{ for all } t \in [0,1])).$$

where $n \in \mathbb{N}^*$ is given by the hypothesis (h3).

In order to verify that (i2') holds we can choose

$$c_j = (j + j_0 + 1/2)/n, \quad \text{for all } j \in \mathbb{N}.$$

Thus, $\lim_{j \to \infty} c_j = \infty$ and it remains to check that $\Phi(S) \cap \{c_j; \ j \in \mathbb{N}\} = \emptyset$. Suppose contrary, that there exist $j \in \mathbb{N}$ and $(x,\lambda) \in S$ such that $c_j = \Phi(x,\lambda)$. Then

$$\inf\{\|x(t)\| ; \ t \in [0,1]\} > r,$$

otherwise, by the construction of $R$, we should have $\sup\{\|x(t)\| ; \ t \in [0,1]\} \leq R$ and, consequently, $\Phi(x,\lambda) < (j_0 + 1/2)/n \leq c_j$ which contradicts $\Phi(x,\lambda) = c_j$.

We define the real function $\theta : [0,1] \to \mathbb{R}$ in the following way

$$\begin{align*}
\theta(0) &= \arg x(0), \\
\theta(t^+_k) &= \theta(t_k) + \arg x(t^+_k) - \arg x(t_k), \text{ for } k \in \overline{1,m}, \\
\theta(t) &\in \text{Arg}x(t), \text{ for } t \in [0,1]
\end{align*}$$

and $\theta$ is continuous at 0 and on each interval $[t_k, t_{k+1}]$, $0 \leq k \leq m$. Clearly, $\theta$ is uniquely determinate by the above properties, moreover, since $x$ is absolutely continuous on each interval $[t_k, t_{k+1}]$, $0 \leq k \leq m$, as a solution of (2) for some $\lambda \in [0,1]$, we also have that $\theta$ is absolutely continuous on each
Since $x(0) = x(1)$, there is an integer $i$ such that $	heta(1) - \theta(0) = 2\pi i$. Hence

\[
(j + j_0 + 1/2)/n = \Phi(x, \lambda) = \left| i + \sum_{k=1}^{m} [\arg x(t_k) - \arg x(t_k^+) \right]/2\pi
\]
or, equivalently,

\[
\frac{n}{\pi} \sum_{k=1}^{m} [\arg x(t_k) - \arg x(t_k^+)] = \pm (2j + 2j_0 - 2ni + 1)
\]
which contradicts the assumption (h3).

Thus, we have proved that (i2') holds.

**Check of (i3'):** Let $x_0 \in K$ be the null function. Then

\[
S(x_0) = \{x \in K; \mu H(x, 0) = x, \text{ for some } \mu \in [0, 1]\}.
\]

Now, by the definition of $H$, we have that each $x \in S(x_0)$ verifies

\[
\begin{cases}
  x'(t) = \mu J \nabla V(x(t)) & \text{for a.e. } t \in [0, 1], \\
  x(t_k^+) = x(t_k^-) & \text{for } k \in \{1, m\}, \\
  x(0) = x(1).
\end{cases}
\]

for some $\mu \in [0, 1]$. Hence

\[
\langle x'(t), \nabla V(x(t)) \rangle = 0 \quad \text{for a.e. } t \in [0, 1],
\]
or, equivalently,

\[
[V(x(t))]' = 0 \quad \text{for a.e. } t \in [0, 1]
\]
Tacking into account that $V(x(t_k^+)) = V(0)$ and that the function $V(x(\cdot))$ is absolutely continuous on $[0, 1]$ except, eventually, the points $t_1, t_2, \ldots, t_m$ in which it is left continuous and admits right limits, we deduce that $V(x(t)) = V(0)$ for all $t \in [0, 1]$. Since $V$ is a proper function on $\mathbb{R}^2$ (see the proof of Lemma 3), we can conclude that $S(x_0)$ is bounded.

So, (i3') also holds, and the theorem is completely proved. □
Remark 4 The main difference between the assumption (h3) and the correspondent assumptions on impulses from [2] is that (h3) allows the impulsive effects to compensate each other. For example, let us consider \( m = 2, \psi^1, \) respectively \( \psi^2, \) one of the continuous branch of the multivalued complex function \( z \rightarrow z^{3/2}, \) respectively \( z \rightarrow z^{1/2}, \) (we have identified \( \mathbb{R}^2 \) with \( \mathbb{C} \)). Then, (h3) is verified for \( n = 1 \) and \( r = 1 \) while the condition on impulses from [2, Example 3] is not satisfied.

Unfortunately, the technique used in this paper can not be extended to other planar differential systems different from a perturbed Hamiltonian. The difficulty arises from the nonuniform rotation around the origin of the solutions for such systems. More precisely, even if (h3) is satisfied for a well chosen \( n \) (in order to agree with the symmetry of the system), the impulses may move integral curves from high angular speed zones to the low ones and, consequently, the functional \( \Phi \) may not be proper anymore.

References


