PERIODIC SOLUTIONS OF SUPERLINEAR IMPULSIVE DIFFERENTIAL SYSTEMS

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ABSTRACT: We develop continuation technique to obtain periodic solutions for superlinear planar differential systems of first order with impulses. Our approach was inspired by some works by Capietto, Mawhin and Zanolin in analogous problems without impulses and uses instead of Brouwer degree the much more elementary notion of essential map in the sense of fixed point theory.

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1. INTRODUCTION

In this paper we study the existence of periodic solutions for planar impulsive differential systems of first order

$$\begin{cases} x' = f(t,x) \qquad \text{for a.e. } t \in [0,1], \\ x(t_k^+) = \psi^k(x(t_k)) \qquad \text{for } k = 1, ..., m, \\ x(0) = x(1), \end{cases} \tag{1.1}$$

where the points $t_k,\ k = 1, ..., m$ ($m \in \mathbf{N}^*$), are fixed and such that $0 < t_1 < ... < t_m < 1$, and we assume

(h1) $f : [0,1] \times \mathbf{R}^2 \to \mathbf{R}^2$ is a L^1-Carathéodory function, while $\psi^k : \mathbf{R}^2 \to \mathbf{R}^2,\ 1 \le k \le m$, are continuous functions.

In particular, we are interested in the solvability of periodic boundary value problems for impulsive second order differential equations of the form

$$\begin{cases} u'' = g(t,u,u') \qquad \text{for a.e. } t \in [0,1], \\ u(t_k^+) \ = \psi_1^k(u(t_k), u'(t_k)) \qquad \text{for } 1 \le k \le m, \\ u'(t_k^+) = \psi_2^k(u(t_k), u'(t_k)) \qquad \text{for } 1 \le k \le m, \\ u(0) = u(1), \quad u'(0) = u'(1). \end{cases} \tag{1.2}$$

The literature on the problems of the above form is quite extensive, see the monographs Bainov & Simeonov [3], Lakshmikantham et al [14] and their references. For applications of the method of lower and upper solutions and monotone iterative techniques we send to [1-2], [9-11], [15-17], while for existence results by means of topological (fixed point, continuation) methods we refer to [7], [12] and [18].

Our approach is based on a Leray-Schauder type continuation principle. Such a principle, namely the topological transversality theorem of Granas (see [6]), has already been applied in the study of boundary value problems for impulsive second order differential equations in [7], in case that one can prove the *a priori* boundedness of all possible solutions of a family of problems connecting (1.2) to a simpler problem corresponding to an essential map. However, there are examples where no *a priori* bounds on solutions exist or can be obtained, see the discussion in [5]. To overcome this difficulty, a variant of the Leray-Schauder continuation principle was introduced in [4] in the frame of degree theory and was applied to periodic boundary value problems without impulses. Recently, the second author has proposed in [18] (see also [19]) a fixed point version of the continuation theorem by Capietto-Mawhin-Zanolin, which does not use the Brouwer degree. For completeness we state this version.

Let X be a real Banach space, $K \subset X$ a convex set and $H : K \times [0,1] \to K$ a completely continuous map. Denote

$$S = \{(x, \lambda) \in K \times [0,1];\ H(x,\lambda) = x\}$$

and for any fixed $x_0 \in K$, let

$$S(x_0) = \{(x,0) \in K \times [0,1];\ (1-\mu)x_0 + \mu H(x,0) = x \text{ for some } \mu \in [0,1]\}.$$

For any set $U \subset X \times [0,1]$ we denote by $U_\lambda = \{x \in X\ ;\ (x,\lambda) \in U\}$, the section of U at λ. Also consider a continuous functional $\Phi : K \times [0,1] \to \mathbf{R}$. Then we have the following theorem [18] :

Theorem 1.1 *Assume there are constants c_- and c_+, $c_- < c_+$, such that, if we denote $U = \Phi^{-1}(]c_-, c_+[)$, the following conditions are satisfied:*

(i1) *$S \cap U$ is bounded;*

(i2) $\Phi(S) \cap \{c_-, c_+\} = \emptyset$;

(i3) *there is $x_0 \in K$ such that $S(x_0)$ is bounded and $S(x_0) \subset U$.*

Then, for each $\lambda \in [0,1]$, there exists at least one fixed point in U_λ for $H(\cdot, \lambda)$.

The proof of Theorem 1.1 is based on an extension of Granas' topological transversality theory to maps $H(\cdot,\lambda)$ having different domains (see [18, Proposition 2]). Next we state a simple consequence of Theorem 1.1 which is more suitable in applications. Before, recall that a map between two metric spaces is said to be *proper* provided that the pre-image of any compact set is also compact.

Corollary 1.2 *Assume*

(i1′) *the restriction of* Φ *to* S *is proper*;

(i2′) Φ *is lower bounded on* S *and there is a sequence* (c_j) *of real numbers such that* $c_j \to \infty$ *and* $c_j \notin \Phi(S)$ *for all* $j \in \mathbf{N}$;

(i3′) *there is* $x_0 \in K$ *such that* $S(x_0)$ *is bounded.*

Then, for each $\lambda \in [0,1]$, *there exists at least one fixed point in* K *for* $H(\cdot,\lambda)$.

Corollary 1.2 was applied in [18] to solve (1.2) under the assumption that the nonlinear function g satisfies a linear growth condition in the last two arguments. The purpose of this paper is to establish existence results for (1.1) and (1.2) in the superlinear case. The proofs will be achieved by means of Corollary 1.2 and of some ideas from [4] and [18] adapted to the present setting.

The functional Φ will be that introduced in [4] as a modification of the classical map which counts the number of rotations around the origin of the continuous integral curves of a planar system (see [13]). The main impediment we have to overcome when we work with impulses is that on however large discontinuous solutions of such problems, the values of Φ may be no integers. Nevertheless, we can find sufficient conditions on the functions ψ^k for that Φ takes values in some disjoint intervals and, by this, condition (i2′) is fulfilled. Our results give such conditions on ψ^k and generalize to impulses the analogous existence theorems from [4]. They are new and complement the existing literature in impulsive differential equations.

We end this introduction with some notations and definitions.

We denote by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ the euclidean scalar product and the norm in $\mathbf{R}^2$. We shall use the following usual functions on $\mathbf{R}^2\setminus\{0\}$:

1) $\arg : \mathbf{R}^2\setminus\{0\} \to [0,2\pi[$ where for $z=(z_1,z_2)$,

$$\arg(z_1,z_2) = \begin{cases} \arctan\left(\frac{z_2}{z_1}\right) & \text{if } z_1>0 \text{ and } z_2>0; \\ \pi/2 & \text{if } z_1=0 \text{ and } z_2>0; \\ \arctan\left(\frac{z_2}{z_1}\right)+\pi & \text{if } z_1<0; \\ 3\pi/2 & \text{if } z_1=0 \text{ and } z_2<0; \\ \arctan\left(\frac{z_2}{z_1}\right)+2\pi & \text{if } z_1>0 \text{ and } z_2<0. \end{cases}$$

2) $\quad \mathrm{Arg} : \mathbf{R}^2 \setminus \{0\} \to 2^{\mathbf{R}}$, $\mathrm{Arg}\, z = \{\arg z \ + 2k\pi;\ k \in \mathbf{Z}\ \}$.

For any real function u we denote by $u^- = -\min\{0, u\}$ its negative part, by $u^+ = \max\{0, u\}$ its positive part and if $u \in L^1(0,1)$, by $\overline{u} = \int_0^1 u(t)dt$, its mean value on $[0,1]$.

For a continuous function $Q : \mathbf{R}^2 \to \mathbf{R}_+^*$ we denote by

$$\widehat{Q} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{Q(\cos\theta, \sin\theta)} \ ,$$

the integral average of $1/Q$ on the unit circle, and by

$$q = \min\{Q(z);\ z \in \mathbf{R}^2,\ \mid z \mid = 1\},$$

the minimum of Q on the unit circle. In case that Q depends on some index j , i.e. $Q = Q_j$, we write

$$q_j = \min\{Q_j(z);\ z \in \mathbf{R}^2,\ \mid z \mid = 1\}.$$

Finally, we recall that a continuous function $Q : \mathbf{R}^2 \to \mathbf{R}$ is said to be *positively homogeneous of second degree and positive definite,* if

$$Q(tz) = t^2 Q(z) > 0$$

for all $t > 0$ and $z \in \mathbf{R}^2 \setminus \{0\}$.

2. AN EXISTENCE PRINCIPLE

The operator form of (1.1)

We are looking for solutions of (1.1) in the following space of functions

$$\begin{aligned} C_T = \ & \{x : [0,1] \to \mathbf{R}^2; x \text{ is everywhere continuous except} \\ & \quad \text{at most points } (t_k)_{1 \le k \le m} \text{ of discontinuity of first type ,} \\ & \quad \text{at which } x \text{ is left continuous}\}, \end{aligned}$$

endowed with the usual $C-$norm, $\parallel x \parallel = \sup\{\mid x(t) \mid;\ t \in [0,1]\}$. Notice that C_T can be identified with the Banach space $\prod_{k=0}^{m} C[t_k, t_{k+1}]$ $(t_0 = 0,\ t_{m+1} = 1)$. Thus, C_T is a Banach space too. Also denote $L^1 = L^1(0,1;\mathbf{R}^2)$ and

$$\begin{aligned} W_p^{1,1} = \ & \{x \in C_T;\ x \text{ is absolutely continuous on each} \\ & \quad]t_k\ ,\ t_{k+1}[,\ k = 0, 1, \ldots, m \text{ and } x(0) = x(1)\}. \end{aligned}$$

Clearly, if $x \in W_p^{1,1}$, then x belongs to the Sobolev space $W^{1,1}(t_k, t_{k+1}; \mathbf{R}^2)$ for $k = 0, 1, \ldots, m$.

Now we consider the linear map

$$\mathcal{L} : W_p^{1,1} \to L^1 \times (\mathbf{R}^2)^m$$

$$\mathcal{L}(x) = (x', \{x(t_k^+)\}_{1\leq k\leq m}).$$

This map is invertible and to get its inverse

$$\mathcal{L}^{-1} : L^1 \times (\mathbf{R}^2)^m \to C_T$$

we have to solve m initial value problems:

$$\begin{cases} x' = y & \text{for a.e. } t \in [t_k, t_{k+1}], \\ x(t_k) = u_k \end{cases}$$

for $1 \leq k \leq m-1$, and

$$\begin{cases} x' = \widetilde{y} & \text{for a.e. } t \in [t_m, 1 + t_1], \\ x(t_m) = u_m, \end{cases}$$

where $\widetilde{y}(t) = y(t)$ on $[t_m, 1]$, $\widetilde{y}(t) = y(t-1)$ on $[1, 1+t_1]$, $y \in L^1$ and $u = \{u_k\}_{1\leq k\leq m} \in (\mathbf{R}^2)^m$. Thus, the unique solution $x \in C_T$ to $\mathcal{L}(x) = (y, u)$ is the function :

$$\begin{aligned} x(t) &= u_k + \int_{t_k}^{t} y(s)ds \quad \text{for } t_k < t \leq t_{k+1},\ 1 \leq k \leq m-1, \\ x(t) &= u_m + \int_{t_m}^{t} y(s)ds \quad \text{for } t_m < t \leq 1, \\ x(t) &= u_m + \int_{t_m}^{1+t} \widetilde{y}(s)ds \quad \text{for } 0 \leq t \leq t_1. \end{aligned} \tag{2.1}$$

We also define the nonlinear map

$$N : C_T \to L^1 \times (\mathbf{R}^2)^m$$

$$N(x) = (f(\cdot, x), \{\psi^k(x(t_k))\}_{1\leq k\leq m}).$$

Then, under assumption (h1) , N is well-defined, continuous and bounded. Moreover, by (2.1) and Ascoli-Arzela theorem, the map $\mathcal{L}^{-1}N : C_T \to C_T$ is completely continuous.

Thus, (1.1) is equivalent with the following fixed point problem

$$x = \mathcal{L}^{-1}N(x), \quad x \in C_T \tag{2.2}$$

The homotopy H

In order to apply a continuation argument we embed (2.2) into a one-parameter family of equations

$$x = \mathcal{L}^{-1} N_\lambda(x),\ x \in C_T \quad (\lambda \in [0,1]), \tag{2.2$_\lambda$}$$

with N_λ of the form

$$N_\lambda(x) = (F(\cdot, x, \lambda), \{\Psi^k(x(t_k), \lambda)\}_{1 \le k \le m}),$$

where

(H1) $\Psi^k(x, \lambda),\ 1 \le k \le m$, are continuous maps from $\mathbf{R}^2 \times [0,1]$ into $\mathbf{R}^2$ and $\Psi^k(x, 1) = \psi^k(x)$;
$F(t, x, \lambda)$ is a L^1-Carathéodory function and $F(t, x, 1) = f(t, x)$.

(Recall that $F(t, x, \lambda)$ is a L^1-Carathéodory function if $F(\cdot, x, \lambda)$ is measurable for each $(x, \lambda) \in \mathbf{R}^2 \times [0,1]$, $F(t, \cdot, \cdot)$ is continuous for a.e. $t \in [0,1]$ and, for each $r > 0$, there exists $\eta_r \in L^1(0,1)$ such that $\mid F(t, x, \lambda) \mid \le \eta_r(t)$ for $\mid x \mid \le r,\ \lambda \in [0,1]$ and a.e. $t \in [0,1]$).

Under assumption (H1), the homotopy

$$H : C_T \times [0,1] \to C_T, \quad H(x, \lambda) = \mathcal{L}^{-1} N_\lambda(x) \tag{2.3}$$

is completely continuous.

In the fourth section we shall describe two such homotopies connecting $H(\cdot, 1) = \mathcal{L}^{-1}N$ to a much simpler map $H(\cdot, 0) = \mathcal{L}^{-1}N_0$.

The functional Φ

We attach to each L^1-Carathéodory function $f(t, x)$ from $[0,1] \times \mathbf{R}^2$ into $\mathbf{R}^2$, the continuous functional

$$\varphi_f : C_T \to \mathbf{R},$$

$$\varphi_f(x) = \frac{1}{2\pi} \mid \int_0^1 [x_2(t) f_1(t, x(t)) - x_1(t) f_2(t, x(t))] \omega(x(t)) dt \mid,$$

where $x = (x_1, x_2),\ f = (f_1, f_2)$ and $\omega(x) = \min\{1, 1/ \mid x \mid^2\}$.

It is easy to see that on each continuous periodic solution of the system $x' = f(t, x)$ large that $\mid x(t) \mid \ge 1$ for all $t \in [0,1]$, $\varphi_f(x)$ reduces to the winding number around the origin of the curve $\{x(t);\ t \in [0,1]\}$ (see [13]). Hence, on such functions, the values of φ_f are integers. Our first step is to obtain estimates for $\varphi_f(x)$ on large periodic solutions $x \in C_T$ (with possible discontinuities on points $(t_k)_{1\le k\le m}$) to the system $x' = f(t,x)$.

Let $n \in \mathbf{N}^*$ be arbitrary fixed and $x \in C_T$ be any periodic solution of $x' = f(t,x)$ satisfying $\mid x(t) \mid\ge 1$ for all $t \in [0,1]$. For each $k \in \{1,\dots,m\}$ we consider the numbers $\gamma_i^k = \gamma_i^k(x)$,

$$\gamma_i^k = 2\pi i/n + \arg x(t_k) - \arg x(t_k^+), \quad -n \le i \le n,$$

and we denote i_k that index for which

$$\mid \gamma_{i_k}^k \mid = \min\{\mid \gamma_i^k \mid;\ -n \le i \le n\}.$$

Also, we concisely write γ_k instead of $\gamma_{i_k}^k$. Next we define the function $\theta : [0,1] \to \mathbf{R}$ such that:

1) θ is absolutely continuous on $[0,t_1]$ and on each interval $]t_k, t_{k+1}]$, $1 \le k \le m$;
2) $\theta(0) = \arg x(0),\ \theta(t) \in \operatorname{Arg} x(t) \quad$ for $t \in [0,t_1]$;
3) $\theta(t_k^+) = \theta(t_k) - \gamma_k\ ,\ \theta(t) \in \operatorname{Arg} x(t) - \frac{2\pi}{n}\sum_{j=1}^{k} i_j \quad$ for $t \in]t_k, t_{k+1}]$, $1 \le k \le m$.

Since $x(0) = x(1)$, by 3), we have

$$\theta(1) - \theta(0) = 2\pi\nu - \frac{2\pi}{n}\sum_{k=1}^{m} i_k \tag{2.4}$$

for some $\nu \in \mathbf{Z}$.

Lemma 2.1 *Assume $f : [0,1]\times\mathbf{R}^2 \to \mathbf{R}^2$ is a L^1-Carathéodory function and $x \in C_T$ is a periodic solution to $x' = f(t,x)$ such that $\mid x(t) \mid\ge 1$ for all $t \in [0,1]$. Then*

$$\mid \nu - \frac{1}{n}\sum_{k=1}^{m} i_k \mid - \frac{1}{2\pi} \mid \sum_{k=1}^{m}\gamma_k \mid \le \varphi_f(x) \le \mid \nu - \frac{1}{n}\sum_{k=1}^{m} i_k \mid + \frac{1}{2\pi} \mid \sum_{k=1}^{m}\gamma_k \mid . \tag{2.5}$$

Proof. Since

$$\theta'(t) = [x_2'(t)x_1(t) - x_1'(t)x_2(t)]/ \mid x(t) \mid^2,$$

we easily obtain

$$\begin{aligned}
\varphi_f(x) &= \frac{1}{2\pi} \mid \int_0^1 \theta'(t)dt \mid = \frac{1}{2\pi} \mid \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} \theta'(t)dt \mid \\
&= \frac{1}{2\pi} \mid \sum_{k=0}^{m} (\theta(t_{k+1}) - \theta(t_k^+)) \mid = \frac{1}{2\pi} \mid \theta(1) - \theta(0) + \sum_{k=1}^{m} (\theta(t_k) - \theta(t_k^+)) \mid \\
&= \mid \nu - \frac{1}{n}\sum_{k=1}^{m} i_k + \frac{1}{2\pi}\sum_{k=1}^{m}\gamma_k \mid,
\end{aligned}$$

whence (2.5) follows immediately. ■

Notice that in case x is continuous on $[0,1]$, we have $i_k = 0$ and $\gamma_k = 0$ for all $1 \leq k \leq m$; consequently, $\varphi_f(x) =\mid \nu \mid$ where $\theta(1) - \theta(0) = 2\pi\nu,\ \nu \in \mathbf{Z}$. Thus, Lemma 2.1 extends to discontinuous functions of class C_T, Proposition 1 from [4].

Now, if $F(t,x,\lambda)$ is a L^1-Carathéodory function, we denote by Φ the continuous functional

$$\Phi : C_T \times [0,1] \to \mathbf{R}, \quad \Phi(x,\lambda) = \varphi_{F(\cdot,\cdot,\lambda)}(x). \tag{2.6}$$

The existence principle

Let $n \in \mathbf{N}^*$ be fixed. For each $k \in \{1,\ldots,m\}$ define

$$\Gamma_i^k(x,\lambda) = 2\pi i/n + \arg x - \arg \Psi^k(x,\lambda)$$

for all $\lambda \in [0,1]$ and $x \in \mathbf{R}^2 \backslash \{0\}$ with $\Psi^k(x,\lambda) \neq 0\ \ (-n \leq i \leq n)$. Denote $i_k = i_k(x,\lambda)$ that index for which

$$\mid \Gamma_{i_k}^k(x,\lambda) \mid = \min\{\mid \Gamma_i^k(x,\lambda) \mid;\ -n \leq i \leq n\}$$

and write $\Gamma_k(x,\lambda)$ instead of $\Gamma_{i_k}^k(x,\lambda)$, for simplicity.

Now, let us list for convenience, the following conditions:

(H2) there exist $n \in \mathbf{N}^*$ and $R \geq 1$ such that

$$\mid \sum_{k=1}^{m} \Gamma_k(x_k,\lambda) \mid < \pi/n \tag{2.7}$$

for each $\lambda \in [0,1]$ and all $(x_k)_{1\leq k\leq m} \in (\mathbf{R}^2)^m$ satisfying $\mid x_k \mid \geq R$ and $\mid \Psi^k(x_k,\lambda) \mid \geq R$ for every $k \in \{1,\ldots,m\}$.

(H3) for every $R_* > 0$ there exists $R^* \geq R_*$ such that for each $(x,\lambda) \in S$ with $\inf\{\mid x(t) \mid;\ t \in [0,1]\} \leq R_*$, one has $\parallel x \parallel \leq R^*$.

(H4) for every $j \in \mathbf{N}$, there exists $R_j > 0$ such that if $(x,\lambda) \in S$ and

$$\Phi(x,\lambda) \in](j-1/2)/n, (j+1/2)/n[\ ,$$

then $\inf\{\mid x(t) \mid;\ t \in [0,1]\} \leq R_j$.

(H5) there exists $x_0 \in C_T$ such that $S(x_0)$ is bounded.

We can now state our general existence principle.

Theorem 2.2 *Suppose* (H1)-(H5) *hold. Then* (1.1) *has at least one solution.*

Proof. We apply Corollary 1.2 with $X = K = C_T$, the homotopy H given by (2.3) and the functional Φ defined by (2.6).

If $(x, \lambda) \in S$ and $| x(t) | \geq R$ for all t, then $\gamma_i^k(x) = \Gamma_i^k(x(t_k), \lambda)$ for $1 \leq k \leq m$ and $-n \leq i \leq n$. Hence $\gamma_k(x) = \Gamma_k(x(t_k), \lambda)$ for $1 \leq k \leq m$. Thus, by (H2) and (2.5), there exists $l \in \mathbf{N}$, namely $l = | n\nu - \sum_{k=1}^{m} i_k |$, such that $\Phi(x, \lambda) \in](l - 1/2)/n, (l + 1/2)/n[$. Next, by (H3) and (H4), we deduce that $S \cap \Phi^{-1}(](j-1/2)/n, (j+1/2)/n[)$ is bounded for each $j \in \mathbf{N}$. This implies, as we can easily see, that the restriction of Φ to S is proper, and so (i1$'$) is satisfied.

Clearly (i2$'$) holds with zero as lower bound of Φ on S and with $c_j = (j + 1/2)/n,\ j \geq j_0$; where j_0 is large that $(j_0 + 1/2)/n > \Phi(x, \lambda)$ for all $(x, \lambda) \in S$ with $\| x \| < R$.

Finally, (H5) is precisely (i3$'$) and thus we may apply Corollary 1.2. ∎

3. A PRIORI ESTIMATES OF SOLUTIONS

In this preparatory section we interrupt the study of impulsive differential systems; we deal here with usual (absolutely continuous) solutions for differential systems without impulses. The results will be used in the next section to verify (H3) and (H5).

The following lemma is a refinement of Proposition 3 from [4] and, roughly speaking, applies to solutions of both periodic and initial value problems.

Lemma 3.1 *Let $\chi \in \{-1, +1\}$, $V \in C^1(\mathbf{R}^2)$ with $| V(x) | \to \infty$ as $| x | \to \infty$, $f : [0, 1] \times \mathbf{R}^2 \to \mathbf{R}^2$ a Carathéodory function and $\sigma \in L^1(0, 1; \mathbf{R}_+)$. Suppose that there is $R' > 0$ such that $V(x) \neq 0$ for $| x | \geq R'$ and*

$$\chi \langle \operatorname{grad} V(x), f(t, x) \rangle / V(x) \leq \sigma(t) \tag{3.1}$$

for a.e. $t \in [0, 1]$ and every $x \in \mathbf{R}^2$ with $| x | \geq R'$. Then, for each $r_1 \geq 0$, there exists $r_2 \geq r_1$ (depending only on r_1, V and σ) such that, for each (absolutely continuous) solution $x(t)$ on some interval $[a, b] \subset [0, 1]$ of $x' = f(t, x)$, for which there are $\tau_0, \tau_1 \in [a, b]$ with

$$\chi(\tau_0 - \tau_1) > 0,\ | x(\tau_1) | \leq r_1,\ | x(\tau_0) | = \max\{| x(t) |;\ t \in [a, b]\}, \tag{3.2}$$

one has

$$\max\{| x(t) |;\ t \in [a, b]\} \leq r_2. \tag{3.3}$$

Proof. We follow the same reasoning as in the proof of Proposition 3 from [4]. Define $W(x) = \log | V(x) |$ for $| x | \geq R'$. Since $| V(x) | \to \infty$ $| x | \to \infty$, one has

$$W(x) \to \infty \quad \text{as} \quad | x | \to \infty\ . \tag{3.4}$$

On the other hand,

$$\operatorname{grad} W(x) = V(x)^{-1} \operatorname{grad} V(x) \quad \text{for } \mid x \mid \geq R'. \tag{3.5}$$

Further, let $c_1 = \max\{r_1, R'\}$ and let $x(t)$ be a solution to $x' = f(t,x)$ on $[a,b]$ satisfying (3.2). If $\mid x(\tau_0) \mid > c_1$, then there is $\tau_2 \in [a,b]$ such that

$$\chi(\tau_0 - \tau_2) > 0, \ \mid x(\tau_2) \mid = c_1, \ \mid x(t) \mid \geq c_1 \ \text{ for } t \in [\tau_2, \tau_0] \text{ (or } t \in [\tau_0, \tau_2]).$$

Now, denote $w(t) = W(x(t))$. Since from (3.5),

$$w'(t) = \langle \operatorname{grad} V(x(t)), f(t, x(t)) \rangle / V(x(t)),$$

by (3.1), we obtain

$$\begin{aligned} w(\tau_0) &= w(\tau_2) + \int_{\tau_2}^{\tau_0} w'(t)dt \leq w(\tau_2) + \mid \int_{\tau_2}^{\tau_0} \sigma(t)dt \mid \\ &\leq \max\{W(x); \mid x \mid = c_1\} + \parallel \sigma \parallel_{\mathrm{L}^1(0,1)} = c_2. \end{aligned}$$

From (3.4) it follows that there is $r_2 \geq c_1$ such that $W(x) > c_2$ for $\mid x \mid > r_2$. Hence, $\mid x(\tau_0) \mid \leq r_2$. ∎

We point out that the bound r_2 does not depend on the subinterval $[a,b]$ of $[0,1]$, and is the same for all Carathéodory functions f satisfying (3.1) with fixed V and σ.

Remarks 3.2 (a) If $\chi = +1$ in (3.1), then (3.3) holds for each solution of $x' = f(t,x)$ on $[a,b]$ which satisfies $\mid x(a) \mid \leq r_1$.

(b) If $\chi = -1$ in (3.1), then (3.3) holds for each solution of $x' = f(t,x)$ on $[a,b]$ which satisfies $\mid x(b) \mid \leq r_1$.

(c) If, instead of (3.1), we assume the following inequality

$$\mid \langle \operatorname{grad} V(x), f(t,x) \rangle \mid \leq \sigma(t) \mid V(x) \mid, \tag{3.6}$$

then (3.3) holds for each solution of $x' = f(t,x)$ on $[a,b]$ which satisfies $\mid x(\tau_1) \mid \leq r_1$ for some arbitrary $\tau_1 \in [a,b]$.

Indeed, (3.6) implies that (3.1) holds with both $\chi = \pm 1$ and so, in (3.2), the location of τ_1 with respect to τ_0 is not important.

(d) If, instead of (3.1), we assume the inequality

$$\langle \operatorname{grad} V(x), f(t,x) \rangle \leq \sigma(t) \mid V(x) \mid, \tag{3.7}$$

then (3.3) holds for each solution of $x' = f(t,x)$ on $[a,b]$ which satisfies $x(a) = x(b)$ and $\mid x(\tau_1) \mid \leq r_1$ for some arbitrary $\tau_1 \in [a,b]$.

Indeed, by (3.7), inequality (3.1) is satisfied with $\chi = +1$ if $V(x) > 0$ for all $| x | \geq R'$, and with $\chi = -1$ in case that $V(x) < 0$ for all $| x | \geq R'$ (since $| V(x) | \to \infty$ as $| x | \to \infty$, we may suppose, either $V(x) > 0$ for all $| x | \geq R'$, or $V(x) < 0$ for all $| x | \geq R'$).

Note that the result in Remark 3.2 (d) is essentially that in Proposition 3 from [4]. Therefore, Lemma 3.1 generalizes Proposition 3 [4].

4. MAIN EXISTENCE RESULTS

This section is devoted to the investigation of some applications of Theorem 2.2. We shall give sufficient conditions on f and ψ^k, $1 \le k \le m$, in order that Theorem 2.2 applies.

Let us first list some conditions, part of them already introduced in [4]:

(h2) there exist $r \ge 1$ and $0 \le \delta < \pi$ such that

$$\sum_{k=1}^{m} \min\{\mid 2\pi i + \arg x_k - \arg \psi^k(x_k) \mid;\ i = -1, 0, 1\} \le \delta \tag{4.1}$$

for each $(x_k)_{1\le k\le m} \in (\mathbf{R}^2)^m$ with $\mid x_k \mid \ge r$ and $\mid \psi^k(x_k) \mid \ge r$ for all $1 \le k \le m$.

(h3) $f(t,x) = -Jh(t,x)$, with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $h(t,x) = \operatorname{grad} V(x) + p(t,x)$, where

1^0 $V \in C^1(\mathbf{R}^2)$, $\mid V(x) \mid \to \infty$ as $\mid x \mid \to \infty$, and $p : [0,1] \times \mathbf{R}^2 \to \mathbf{R}^2$ is L^1-Carathéodory;

2^0 there exists $r_0 > 0$ such that $\operatorname{grad} V(x) \ne 0$ for $\mid x \mid \ge r_0$;

3^0 $\limsup_{|x|\to\infty} \langle \operatorname{grad} V(x), -Jp(t,x)\rangle / \mid V(x) \mid \le \alpha(t)$ uniformly a.e. in $t \in [0,1]$, for some $\alpha \in L^1(0,1;\mathbf{R}_+)$.

(h4) there exist a sequence (Q_j) of positively homogeneous functions of second degree and positive definite, and a sequence (β_j) of Lebesgue integrable functions such that

4^0 $\liminf_{|x|\to\infty} [\langle \operatorname{grad} V(x), x\rangle - \langle p(t,x), x\rangle^-] / Q_j(x) \ge \beta_j(t)$ uniformly a.e. in $t \in [0,1]$ and for all $j \in \mathbf{N}$;

5^0 $(\overline{\beta_j} - \delta/q_j)/\widehat{Q_j} \to \infty$ as $j \to \infty$.

Theorem 4.1 *Suppose* (h1)-(h4) *hold and in addition that the functions* ψ^k, $1 \le k \le m$, *are proper. Then* (1.1) *has at least one solution.*

Proof. We apply Theorem 2.2 with $n = 1$,

$$\Psi^k(x,\lambda) = \lambda\psi^k(x) + (1-\lambda)x\ ,\ 1 \le k \le m, \tag{4.2}$$

$$F(t,x,\lambda) = -J(\operatorname{grad} V(x) + \lambda p(t,x)) + (1-\lambda)\rho(x)\ , \tag{4.3}$$

where $\rho(x) = (E(\partial V(x)/\partial x_1), E(\partial V(x)/\partial x_2))$ and $E : \mathbf{R} \to \mathbf{R}$ is continuous, bounded and satisfies $tE(t) < 0$ for all $t \ne 0$.

By the definition of ρ, there is $E_0 > 0$ such that

$$| \rho(x) | \leq E_0 \quad \text{for all } x \in \mathbf{R}^2, \tag{4.4}$$

and also, recalling 2^0,

$$\langle \operatorname{grad} V(x), \rho(x) \rangle < 0 \quad \text{for } \ | x | \geq r_0. \tag{4.5}$$

Check of (H1). Use (h1), (4.2) and (4.3).

Check of (H2). First, remark that since ψ^k are proper, there exists $R \geq r$ such that

$$| x | \geq R \text{ implies } \ | \psi^k(x) | \geq r \text{ for all } 1 \leq k \leq m.$$

Next, by (4.2) and (h2), it follows that

$$| \Gamma_k(x, \lambda) | \leq \min\{| \, 2\pi i + \arg x - \arg \psi^k(x) \, |; \ i = -1, 0, 1\}$$

for each $x \in \mathbf{R}^2$ with $| x | \geq R$ and all $\lambda \in [0,1]$ $(1 \leq k \leq m)$. Therefore,

$$| \sum_{k=1}^{m} \Gamma_k(x_k, \lambda) | \leq \sum_{k=1}^{m} | \Gamma_k(x_k, \lambda) | \leq \delta < \pi$$

for each $\lambda \in [0,1]$ and all $(x_k)_{1 \leq k \leq m} \in (\mathbf{R}^2)^m$ with $| x_k | \geq R, \ 1 \leq k \leq m$.

Check of (H3). From (4.3), (4.5) and 3^0, it follows that there exists $R' \geq r_0$, large enough that $V(x)$ preserves the same sign for $| x | \geq R'$, such that

$$\chi \langle \operatorname{grad} V(x), F(t, x, \lambda) \rangle / V(x) \leq \alpha(t) + 1 = \sigma(t)$$

for a.e. $t \in [0,1]$, all $x \in \mathbf{R}^2$ with $| x | \geq R'$ and all $\lambda \in [0,1]$, where $\chi = \operatorname{sign} V(x)$. Consequently, we may apply Lemma 3.1 on any subinterval of $[0,1]$.

Let $R_* > 0$ and $(x, \lambda) \in S$ such that

$$\inf\{| \, x(t) \, |; \ t \in [0,1]\} \leq R_*.$$

Suppose that this infimum is achieved on $]t_k, t_{k+1}]$ for some $k, \ 0 \leq k \leq m$. Then, there is $\xi \in [t_k, t_{k+1}]$ such that $| \, x(\xi^+) \, | \leq R_*$. Further, we distinguish the cases $\chi = +1$ and $\chi = -1$. First, assume that $\chi = +1$. Then, by Lemma 3.1, Remark 3.2 (a), there is a number $R_0^* \geq R_*$ depending only on R_*, such that

$$\sup\{| \, x(t) \, |; \ t \in]\xi, t_{k+1}]\} \leq R_0^*.$$

In particular, one has $| \, x(t_{k+1}) \, | \leq R_0^*$ and, by the continuity of $\psi^j, \ 1 \leq j \leq m$, we can find a number $R_*^1 \geq 0$ depending only on R_0^*, such that $| \, \Psi^{k+1}(x(t_{k+1}), \lambda) \, | \leq R_*^1$. Hence,

$$\inf\{| \, x(t) \, |; \ t \in]t_{k+1}, t_{k+2}]\} \leq R_*^1.$$

Next we apply the same reasoning for the interval $]t_{k+1}, t_{k+2}]$ and get $R_1^* \geq R_*^1,\ R_1^* \geq R_0^*$ and $R_*^2 \geq 0$ such that

$$\sup\{\mid x(t) \mid;\ t \in]t_{k+1}, t_{k+2}]\} \leq R_1^*,$$

$$\mid \Psi^{k+2}(x(t_{k+2}), \lambda) \mid \leq R_*^2.$$

Then we apply successively the above arguments to the intervals $]t_{k+2}, t_{k+3}], \ldots,]t_m, 1]$, $[0, t_1],]t_1, t_2], \ldots,]t_{k-1}, t_k]$ and $]t_k, \xi]$, using that $x(1) = x(0)$, and we obtain two systems of numbers

$$R_* \leq R_0^* \leq R_1^* \leq \ldots \leq R_{m+1}^*,$$

$$R_*^1,\ R_*^2,\ \ldots,\ R_*^{m+1}.$$

It is clear that $R^* = R_{m+1}^*$ fulfills (H3).

In case that $\chi = -1$, we shall apply Lemma 3.1, Remark 3.2 (b), to the intervals $]t_k, \xi],]t_{k-1}, t_k], \ldots,]t_1, t_2], [0, t_1],]t_m, 1],]t_{m-1}, t_m], \ldots,]t_{k+1}, t_{k+2}]$ and $]\xi, t_{k+1}]$, in this order. Here we essentially use the hypothesis that ψ^j are proper, as we explain in what follows. First, remark that, if $0 \leq \delta < \pi$, then for each $M_0 > 0$, there is $M > 0$ such that

$$\mid \mu z_1 + (1 - \mu) z_2 \mid > M_0 \quad \text{for all } \mu \in [0, 1],$$

whenever $z_1, z_2 \in \mathbf{R}^2,\ \mid z_1 \mid > M,\ \mid z_2 \mid > M$ and

$$\arccos \frac{\mid z_1 \mid^2 + \mid z_2 \mid^2 - \mid z_1 - z_2 \mid^2}{2 \mid z_1 \mid\mid z_2 \mid} \leq \delta.$$

Returning now to our task, by Lemma 3.1, Remark 3.2 (b), we get $R_0^* \geq R_*$ such that

$$\sup\{\mid x(t) \mid;\ t \in]t_k, \xi]\} \leq R_0^*.$$

In particular, $\mid x(t_k^+) \mid \leq R_0^*$, that is

$$\mid \lambda \psi^k(x(t_k)) + (1 - \lambda) x(t_k) \mid \leq R_0^*.$$

This implies, by the above remark (with $M_0 = R_0^*$) and (4.1), that

$$\mid \psi^k(x(t_k)) \mid \leq M \text{ or } \mid x(t_k) \mid \leq M,$$

for some $M,\ M \geq r$. Since ψ^k is proper, these yield, in both cases

$$\mid x(t_k) \mid \leq R_*^1,$$

for some $R_*^1 \geq M$. Further we apply the same reasoning on each of the next intervals, successively, and we obtain, as in previous case, that (H3) is fulfilled with $R^* = R_{m+1}^*$.

Check of (H4). By 5^0, passing possibly to a subsequence, we may suppose that for each $j \in \mathbf{N}$,

$$(\overline{\beta_j} - \delta/q_j)/(2\pi\widehat{Q_j}) \geq j+2.$$

Then, we can choose $\epsilon_j > 0$ such that

$$(\overline{\beta_j} - \epsilon_j - \delta/q_j)/(2\pi\widehat{Q_j}) \geq j+1. \tag{4.6}$$

From 4^0 and the Carathéodory conditions on $p(t,x)$ it follows that, for each $j \in \mathbf{N}$, there is $\eta_j \in L^1(0,1;\mathbf{R}_+)$ such that

$$\langle \operatorname{grad} V(x), x\rangle - \langle p(t,x), x\rangle^- \geq (\beta_j(t) - \epsilon_j)Q_j(x) - \eta_j(t) \mid x \mid,$$

for all $x \in \mathbf{R}^2$ and a.e. $t \in [0,1]$. Then, for $x \neq 0$, $\lambda \in [0,1]$ and a.e. $t \in [0,1]$, we have

$$\begin{aligned}&[\langle \operatorname{grad} V(x), x\rangle + \lambda\langle p(t,x), x\rangle - (1-\lambda)\langle \rho(x), Jx\rangle]/ \mid x \mid \geq \\ &\geq (\beta_j(t) - \epsilon_j) \mid x \mid Q_j(x/ \mid x \mid) - E_0 - \eta_j(t)\ .\end{aligned} \tag{4.7}$$

Now, for each $j \in \mathbf{N}$ consider $R_j \geq r$. Let $(x,\lambda) \in S$ such that $\inf\{\mid x(t) \mid;\ t \in [0,1]\} > R_j$. From (4.7), using (4.3) and integrating on $[0,1]$, we deduce

$$\int_0^1 \frac{x_2(t)x_1'(t) - x_1(t)x_2'(t)}{\mid x(t) \mid^2 Q_j(x(t)/ \mid x(t) \mid)} dt \geq \overline{\beta_j} - \epsilon_j - \frac{E_0 + \overline{\eta_j}}{R_j q_j}. \tag{4.8}$$

Recalling now the definition of the function $\theta(t)$ attached to x and using the $2\pi-$periodicity in θ of $Q_j(\cos\theta, \sin\theta)$, we can compute

$$\begin{aligned}\int_0^1 \frac{x_2(t)x_1'(t) - x_1(t)x_2'(t)}{\mid x(t) \mid^2 Q_j(x(t)/ \mid x(t) \mid)} dt &= \int_0^1 \theta'(t)dt/Q_j(\cos\theta(t), \sin\theta(t)) \\
&= \sum_{k=0}^m \int_{t_k}^{t_{k+1}} \theta'(t)dt/Q_j(\cos\theta(t), \sin\theta(t)) \\
&= \sum_{k=0}^m \int_{\theta(t_k^+)}^{\theta(t_{k+1})} d\theta/Q_j(\cos\theta, \sin\theta) \\
&= (\theta(1) - \theta(0))\widehat{Q_j} - \sum_{k=1}^m \int_{\theta(t_k)}^{\theta(t_k^+)} d\theta/Q_j(\cos\theta, \sin\theta) \\
&= 2\pi\widehat{Q_j}\frac{1}{2\pi}[\theta(1) - \theta(0) + \sum_{k=1}^m (\theta(t_k) - \theta(t_k^+))] + \\
&+ \sum_{k=1}^m \int_{\theta(t_k)}^{\theta(t_k^+)} [\widehat{Q_j} - 1/Q_j(\cos\theta, \sin\theta)]d\theta\end{aligned}$$

$$= \quad 2\pi\widehat{Q_j}\chi\Phi(x,\lambda)+\sum_{k=1}^{m}\int_{\theta(t_k)}^{\theta(t_k^+)}[\widehat{Q_j}-1/Q_j(\cos\theta,\sin\theta)]d\theta \ ,$$

where $\chi \in \{-1,+1\}$ (recalling also the computation of $\Phi(x,\lambda)$ in the proof of Lemma 2.1). Replacing the above relation in (4.8) we get

$$\begin{aligned} 2\pi\widehat{Q_j}\chi\Phi(x,\lambda) \ &\geq \ \overline{\beta_j}-\epsilon_j-(E_0+\overline{\eta_j})/(R_jq_j)- \\ &- \ \sum_{k=1}^{m}\left|\int_{\theta(t_k)}^{\theta(t_k^+)}[\widehat{Q_j}-1/Q_j(\cos\theta,\sin\theta)]d\theta\right| . \end{aligned}$$

Since $0 < \widehat{Q_j} \leq 1/q_j$ and $0 < 1/Q_j(\cos\theta,\sin\theta) \leq 1/q_j$, we have $|\widehat{Q_j}-1/Q_j(\cos\theta,\sin\theta)| < 1/q_j$. Then,

$$\begin{aligned} 2\pi\widehat{Q_j}\chi\Phi(x,\lambda) \ &\geq \ \overline{\beta_j}-\epsilon_j-(E_0+\overline{\eta_j})/(R_jq_j)- \\ &- \ \frac{1}{q_j}\sum_{k=1}^{m}|\ \theta(t_k^+)-\theta(t_k)\ | \ , \end{aligned}$$

and since

$$|\ \theta(t_k^+)-\theta(t_k)\ |=|\ \Gamma_k(x,\lambda)\ | \quad \text{for all } 1\leq k\leq m,$$

we deduce

$$\begin{aligned} \chi\Phi(x,\lambda) \ &\geq \ (\overline{\beta_j}-\epsilon_j-\delta/q_j)/(2\pi\widehat{Q_j})- \\ &- \ (E_0+\overline{\eta_j})/(2\pi\widehat{Q_j}R_jq_j) \ . \end{aligned}$$

Now, using (4.6) and choosing $R_j \geq \max\{r,(E_0+\overline{\eta_j})/(\pi\widehat{Q_j}q_j)\}$, we deduce that

$$\chi\Phi(x,\lambda)\geq j+1-1/2=j+1/2.$$

Consequently, $\chi=+1$ and therefore,

$$\Phi(x,\lambda)\geq j+1/2.$$

Thus, we have obtained that for each $(x,\lambda)\in S$ with $\inf\{|\ x(t)\ |;\ t\in[0,1]\}>R_j$, we have $\Phi(x,\lambda)\geq j+1/2$. Hence, if $(x,\lambda)\in S$ and $\Phi(x,\lambda)\in]j-1/2,j+1/2[$, then $\inf\{|\ x(t)\ |;\ t\in[0,1]\}\leq R_j$ and so, (H4) is fulfilled.

Check of (H5). Let $x_0=0$. Then, $S(0)$ reduces to the set of all periodic solutions of the autonomous systems

$$x'=\mu(-J\,\mathrm{grad}\,V(x)+\rho(x)), \quad \mu\in[0,1]. \tag{4.9$_\mu$}$$

From (4.5) we deduce that $V(x)$ is a guiding function for (4.9_μ) (see [8, p.82]). Consequently, the periodic solutions of (4.9_μ) are bounded uniformly with respect to μ.

Thus, the assumptions (H1)-(H5) are fulfilled and we may apply Theorem 2.2. So, the proof of Theorem 4.1 is complete. ■

Remark 4.2 We may replace hypothesis 4^0 by

$4'$ $\limsup_{|x|\to\infty}[\langle \operatorname{grad} V(x), x\rangle + \langle p(t,x), x\rangle^+]/Q_j(x) \le -\beta_j(t)$
uniformly a.e. in $t \in [0,1]$ and for all $j \in \mathbf{N}$;

and the conclusion of Theorem 4.1 remains true (see the similar Remark 5 from [4]).

Note that Theorem 4.1 generalizes for impulses Theorem 4 in [4]. Consequently, Theorem 4.1 assures the existence of solutions for the equations and systems from Examples 4, 5 and 6 in [4], even in case that impulsive effects subject to (h2) and to the properness condition are considered.

Our next goal is to replace (h2) by a more general condition depending on an arbitrary $n \in \mathbf{N}^*$. We shall do this requiring a little stronger condition on f. Instead, we have not to suppose that ψ^k are proper.

Let n be an arbitrary fixed integer, $n \ge 1$. We shall refer to (h2$'$) as to (h2) where $\delta < \pi/n$ and instead of (4.1) we require

$$\sum_{k=1}^{m} \min\{|\ 2\pi i/n + \arg x_k - \arg \psi^k(x_k)\ |;\ -n \le i \le n\} \le \delta. \tag{4.10}$$

Also, we shall refer to (h3$'$) as to (h3), where 3^0 is replaced by

$3'$ $\limsup_{|x|\to\infty}\langle \operatorname{grad} V(x), -Jp(t,x)\rangle/V(x) \le \alpha(t)$,

and to (h4$'$) as to (h4) where, in addition, the functions $Q_j(\cos\theta, \sin\theta)$ are assumed to be $2\pi/n-$periodic.

Theorem 4.3 *Suppose* (h1), (h2$'$), (h3$'$) *and* (h4$'$) *hold. Then* (1.1) *has at least one solution.*

Proof. This time we apply Theorem 2.2 with

$$\begin{aligned} \Psi^k(x,\lambda) &= \lambda\psi^k(x) \quad , \quad 1 \le k \le m, & (4.11)\\ F(t,x,\lambda) &= -J(\operatorname{grad} V(x) + \lambda p(t,x)). & (4.12) \end{aligned}$$

Check of (H1). It is immediate by (h1), 1^0, (4.11) and (4.12).

Check of (H2). Since, by (4.11), $\arg \Psi^k(x,\lambda) = \arg \psi^k(x)$ for all $\lambda \in]0,1]$ and $x \in \mathbf{R}^2$ with $\psi^k(x) \ne 0$, we have $\Gamma_i^k(x,\lambda) = 2\pi i/n + \arg x - \arg \psi^k(x)$ for $-n \le i \le n$. Thus, (h2$'$) clearly implies (H2).

Check of (H3). From (4.12) and $3'$, it follows that there exists $R' \ge r_0$, large enough that $V(x) \ne 0$ for $|\ x\ | \ge R'$, such that

$$\langle \operatorname{grad} V(x), F(t,x,\lambda)\rangle/V(x) \le \alpha(t) + 1$$

for a.e. $t \in [0,1]$; all $x \in \mathbf{R}^2$ with $\mid x \mid \geq R'$ and all $\lambda \in [0,1]$. Further we argue as in the proof of Theorem 4.1, by using Lemma 3.1, Remark 3.2 (a).

Check of (H4). We use the same reasoning as in the proof of Theorem 4.1. Here, by (2.4) and the $2\pi/n-$periodicity of $Q_j(\cos\theta, \sin\theta)$, we equally have that

$$\int_{\theta(0)}^{\theta(1)} d\theta / Q_j(\cos\theta, \sin\theta) = (\theta(1) - \theta(0))\widehat{Q_j}.$$

Check of (H5). Let $x_0 = 0$. Then $S(0)$ is the set of all solutions $x \in C_T$ to

$$\begin{cases} x' = \mu[-J \operatorname{grad} V(x)] \quad \text{a.e. } t \in [0,1], \\ x(0) = x(1), \\ x(t_k^+) = 0, \ 1 \leq k \leq m \quad (\mu \in [0,1]). \end{cases}$$

To prove its boundedness it is sufficient to apply Lemma 3.1, Remark 3.2 (a) (with $\chi = +1$ and $\sigma(t) \equiv 0$) on each interval $[t_k, t_{k+1}],\ 1 \leq k \leq m$, and on $[0, t_1]$.

Thus we can apply Theorem 2.2 and the proof is complete. ■

5. APPLICATIONS

In this section we shall apply our results to some well-known equations with superlinear growth. The Examples we deal here with, have already been presented in [4] in the case without impulses.

EXAMPLE 1. Consider the second order scalar equation with impulses

$$\begin{cases} u'' + g(u) = q(t, u, u') \qquad \text{for a.e. } t \in [0,1], \\ u(t_k^+) \ = \psi_1^k(u(t_k), u'(t_k)) \qquad \text{for } 1 \leq k \leq m, \\ u'(t_k^+) = \psi_2^k(u(t_k), u'(t_k)) \qquad \text{for } 1 \leq k \leq m, \\ u(0) = u(1), \quad u'(0) = u'(1). \end{cases} \tag{5.1}$$

where q is a Carathéodory function, $g : \mathbf{R} \to \mathbf{R}$ and $\psi_1^k,\ \psi_2^k : \mathbf{R}^2 \to \mathbf{R}, 1 \leq k \leq m$, are continuous. We suppose that

$$\lim_{|u| \to +\infty} g(u)/u = +\infty$$

and

$$\mid q(t, u, v) \mid \leq A(\mid u \mid + \mid v \mid) + a(t)$$

for all $u, v \in \mathbf{R}$ and a.e. $t \in [0,1]$, with $A \geq 0$ and $a \in L^1(0,1;\ \mathbf{R}_+)$, and we assume that there exist $n \in \{1,2\},\ r \geq 1$ and $0 \leq \delta < 1/2$ such that

$$\sum_{k=1}^{m} \min\{\mid 2\pi i/n + \arg x_k - \arg(\psi_1^k(x_k), \psi_2^k(x_k)) \mid;\ -n \leq i \leq n\} \leq \delta \tag{5.2}$$

for each $(x_k)_{1\le k\le m} \in (\mathbf{R}^2)^m$ with $| \ x_k \ |\ge r$ and $| \ (\psi_1^k(x_k), \psi_2^k(x_k)) \ |\ge r$ for all $1\le k\le m$. Under these assumptions, one can prove that conditions (h1), (h2′), (h3′) and (h4′) are satisfied with (see also [4, p.382])

$$x = (x_1, x_2) = (u, u'), \ h(t, x) = (g(x_1) - q(t, x_1, x_2), x_2),$$

$$V(x) = \int_0^{x_1} g(s)ds \ + \ \frac{1}{2}x_2^2,$$

$$p(t, x) = (-q(t, x_1, x_2), 0)$$

and

$$Q_j(x) = 2j^2x_1^2 + \frac{1}{2}x_2^2 \ , \quad \beta_j = 1 - (a(t)/2j^2).$$

So, we can apply Theorem 4.3 and (5.1) has at least one periodic solution in C_T.

We point out that, even in case $n = 1$, we need $\delta < 1/2$, in order to satisfy 5^0.

Example 2. Consider the second order scalar (Liénard) equation with impulses

$$\begin{cases} u'' + f(u)\ u' + \{u\}^l = q(t, u, u') & \text{for a.e. } t \in [0, 1], \\ u(t_k^+) \ = \psi_1^k(u(t_k), u'(t_k)) & \text{for } 1 \le k \le m, \\ u'(t_k^+) = \psi_2^k(u(t_k), u'(t_k)) & \text{for } 1 \le k \le m, \\ u(0) = u(1), \quad u'(0) = u'(1). \end{cases} \tag{5.3}$$

where q is a Carathéodory function, $f : \mathbf{R} \to \mathbf{R}$, $\psi_1^k, \psi_2^k : \mathbf{R}^2 \to \mathbf{R}$, $1 \le k \le m$, are continuous and $\{u\}^l = u \mid u \mid^{l-1}$, with $l > 1$. If q is bounded and, for

$$F(u) = \int_0^u f(s)ds,$$

we assume

$$\liminf_{|u|\to +\infty} F(u)/u > -\infty,$$

$$\gamma = \limsup_{|u|\to \ +\infty} F^2(u)/ \mid u \mid^{l+1} < 4$$

and if we suppose that there exist $n \in \{1, 2\}$, $r \ge 1$ and $0 \le \delta < 1 - \gamma/4$ such that

$$\sum_{k=1}^m \min\{| \ 2\pi i/n + \arg x_k - \arg(\psi_1^k(x_k), \psi_2^k(x_k)) \ |; \ -n \le i \le n\} \le \delta \tag{5.4}$$

for each $(x_k)_{1\le k\le m} \in (\mathbf{R}^2)^m$ with $| \ x_k \ |\ge r$ and $| \ (\psi_1^k(x_k), \psi_2^k(x_k)) \ |\ge r$ for all $1 \le k \le m$, then (5.3) has at least one periodic solution in C_T. Indeed, conditions (h1), (h2′), (h3′) and (h4′) are fulfilled with (see also [4, pp.383-385])

$$x = (x_1, x_2) = (u, u' + F(u)), \quad h(t, x) = (\{x_1\}^l - q(t, x_1, x_2), x_2 - F(x_1)),$$

$$V(x) = \frac{1}{l+1} \mid x_1 \mid^{l+1} + \frac{1}{2}x_2^2,$$

$$p(t,x) = (-q(t,x_1,x_2), -F(x_1))$$

and

$$Q_j(x) = (j^2/\eta)x_1^2 + \eta x_2^2\ , \quad \beta_j(t) = 1,$$

where η is a fixed real number such that $\delta < \eta < 1-\gamma/4$. Consequently, we can apply Theorem 4.3.

EXAMPLE 3. Consider the (planar) perturbed Hamiltonian system with impulses

$$\begin{cases} x' = J \operatorname{grad} W(x) + q(t,x) & \text{for a.e. } t \in [0,1], \\ x(t_k^+) = \psi^k(x(t_k)) & \text{for } k = 1,2,...,m, \\ x(0) = x(1), \end{cases} \tag{5.5}$$

with $W : \mathbf{R}^2 \to \mathbf{R}$ of class C^1, $q : [0,1] \times \mathbf{R}^2 \to \mathbf{R}^2$ a Carathéodory function and $\psi^k : \mathbf{R}^2 \to \mathbf{R}^2$, $1 \le k \le m$, continuous functions. We suppose that

$$\mid q(t,x) \mid \le \tilde{q}(t), \quad \text{for a.e. } t \in [0,1] \text{ and all } x \in \mathbf{R}^2,$$

with $\tilde{q} \in L^1(0,1;\mathbf{R}_+)$ and there are $A, B \in \mathbf{R}_+$ such that

$$\mid \operatorname{grad} W(x) \mid \le A \mid W(x) \mid + B, \quad \text{for all } x \in \mathbf{R}^2.$$

Assume also the superlinear growth condition

$$\lim_{|x| \to +\infty} \mid \langle \operatorname{grad} W(x), x \rangle \mid / \mid x \mid^2 = +\infty,$$

and that there exist $n \in \mathbf{N}^*$, $0 \le \delta < \pi/n$, and $r \ge 1$ such that

$$\sum_{k=1}^{m} \min\{\mid 2\pi i/n + \arg x_k - \arg(\psi^k(x_k)) \mid;\ -n \le i \le n\} \le \delta \tag{5.6}$$

for each $(x_k)_{1 \le k \le m} \in (\mathbf{R}^2)^m$ with $\mid x_k \mid \ge r$ and $\mid \psi^k(x_k) \mid \ge r$ for all $1 \le k \le m$. Then (5.5) has at least one solution in C_T.

Indeed, we can apply Theorem 4.3 with (see also [4, pp.385-387])

$$V(x) = -W(x),$$

$$p(t,x) = Jq(t,x)$$

and

$$Q_j(x) = j \mid x \mid^2, \quad \beta_j(t) = 1 - (\tilde{q}(t)/j).$$

Finally, we point out that one can reobtain the results from [4], if we consider trivial impulses, i.e. $\psi^k = (\psi_1^k, \psi_2^k)$, $\psi_1^k(x_1,x_2) = x_1$, $\psi_2^k(x_1,x_2) = x_2$. Indeed, in this case, conditions (4.1), (4.10), (5.2), (5.4) and (5.6) hold with $\delta = 0$.

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