1. Let $M$ be a smooth manifold. Prove that the cotangent bundle of $M$, 

$$T^*M = \bigcup_{x \in M} T^*_x M,$$

is also a smooth manifold.

2. (a) Prove that the subset $S \subset (\mathbb{R}^n)^l$ consisting of all linear independent $l$-tuples of vectors in $\mathbb{R}^n$ is open, and the map $\mathbb{R}^l \times S \to \mathbb{R}^n$ defined by

$$((t_1, \ldots, t_l), (v_1, \ldots, v_l)) \mapsto t_1v_1 + \cdots + t_lv_l$$

is a submersion.

(b) Suppose that $M$ is a compact submanifold of $\mathbb{R}^n$. Use part (a) to prove that “almost every” vector subspace $V$ of $\mathbb{R}^n$ of dimension $l$, intersects $M$ transversally. (You may quote the appropriate theorem we proved in class.)

3. Let $f$ and $F$ be a smooth functions on a compact manifold $M$, and let $g$ be a Riemannian metric on $M$.

(a) Let $\phi(t, x)$ be the flow of the vector field $F(\nabla_g f)$. Show that

$$\frac{d}{dt}(f(\phi_x(t))) = F(\phi_x(t))\|\nabla_g f(\phi_x(t))\|^2_g,$$

where $\|\nabla_g f(\phi_x(t))\|^2_g$ is the value of $g(\nabla_g f, \nabla_g f)$ evaluated at $\phi_x(t)$.

(b) Suppose that $a$ and $b$ are regular values of $f$ with $a < b$, and that there are no critical values of $f$ in the interval $[a, b]$. Prove that there is a diffeomorphism of $M$ which maps the submanifold $f^{-1}(a)$ to $f^{-1}(b)$. Hint: use part (a).