1. Problem 8

Solving for intersection, we get $2z^2 = a - 1$. This has real solutions only for $a \geq 1$. We separate into 2 cases: $a = 1$ and $a > 1$. When $a > 1$ the intersection is

$$x^2 + y^2 = (a + 1)/2, z = \pm \sqrt{\frac{a - 1}{2}}$$

which are two circles in the two horizontal planes given by the second equation above. When $a = 1$, the intersection is a single circle $x^2 + y^2 = 1$ in the $x - y$ plane.

Let $f(x, y, z) = x^2 + y^2 - z^2$. The tangent space at $(x, y, z)$ to $x^2 + y^2 - z^2 = 1$ is the kernel of the linear map $df$. This is the subspace spanned by

$$u_1 = \left( y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right), \quad u_2 = \left( 2xy \frac{\partial}{\partial z} + yz \frac{\partial}{\partial x} + xz \frac{\partial}{\partial y} \right)$$

Let $g(x, y, z) = x^2 + y^2 + z^2$. The tangent space at $(x, y, z)$ to $x^2 + y^2 + z^2 = a$ is the kernel of the linear map $dg$. This is the subspace spanned by

$$v_1 = \left( y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right), \quad v_2 = \left( 2xy \frac{\partial}{\partial z} - yz \frac{\partial}{\partial x} - xz \frac{\partial}{\partial y} \right)$$

The sets intersect transversally, if at every point of intersection the above subspaces span $\mathbb{R}^3$. Since $u_1 = v_1$, this means that $u_2 \neq v_2$ and both non-zero. When $a > 1$, we have $z \neq 0$, and so this is indeed the case. So the intersection is transverse. When $a = 1$, we have $z = 0$, and so $u_2 = v_2$ and the intersection is non-transverse.
2. Problem 9

At a point \( v \) in \( V \), the derivative of the diagonal map induces

\[(D\Delta)_v(z) = (z, z) \in T_vV \oplus T_vV\]

The derivative of the graph of the linear map induces

\[(DW)_v(z) = (z, Az) \in T_vV \oplus T_{Av}V\]

Suppose 1 is not an eigenvalue of \( A \). Then the only point of intersection for \( \Delta \) and \( W \) is \((0, 0)\). Moreover, for any non-zero \( z \in T_{(0,0)}(V \oplus V) \) the derivatives are unequal on the second factor.

Conversely, suppose that \( W \) is transverse to \( \Delta \). If there is a point of intersection \((v, v)\) with \( v \neq 0 \) then \( Av = v \). This implies that \((D\Delta)_v(v) = (DA)_v(v)\), so the intersection of the subspaces \((D\Delta)(T_vV)\) and \((DW)(T_vV)\) has dimension at least 1. By a dimension count the subspaces together cannot span \( T_{(v,v)}(V \oplus V) \). So the only point of intersection is \((0, 0)\). By transversality, for all non-zero vectors \( z \) in \( T_{(0,0)}V \) we must have \( Az \neq z \). This is equivalent to saying that \( A \) does not have eigenvalue 1.

3. Problem 10

(a): The map \( q \) factors through the map \( Q : S^{n-1} \to \mathbb{R}P^{n-1} \), which is a two-fold cover of \( \mathbb{R}P^{n-1} \). In particular, \( Q \) is proper and a local diffeomorphism. So given a vector \( x \in V \) of norm 1, it is enough to show that in the local chart \( U \) of \( S^{n-1} \) containing \( x \) on which \( Q \) is a diffeomorphism, the set \( S^{n-1} \cap V \) is a sub-manifold. This is obvious, so we are done. In particular, if \( L_x = \{ w \in \mathbb{R}^n : < w, x > = 0 \} \) then using triviality of the \( T\mathbb{R}^n \) i.e. \( T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \) gives us \( L_x \) as the tangent space to \( S^{n-1} \) at \( x \). The derivative \( dQ \) maps this diffeomorphically onto \( T_{Q(x)}\mathbb{R}P^{n-1} \). If we restrict \( dQ \) to \( V \cap L_x \) in \( T_xS^{n-1} \) then the image is \( T_{q(x)}q(V) \) i.e. the tangent space at \( q(x) \) to the sub-manifold \( q(V) \).

(b): The tangent bundle to \( \mathbb{R}^n \) is trivial i.e. \( T_x\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \). We have \( V \pitchfork W \) iff at every \( x \in V \cap W \) we have \( T_xV + T_xW = T_x\mathbb{R}^n \). Using triviality of the tangent bundle, this is equivalent to

\[ V \pitchfork W \iff V + W = \mathbb{R}^n \iff \dim(V) + \dim(W) - \dim(V \cap W) = n \]

(c): By the reasons mentioned in (a), it is enough to check that \((S^{n-1} \cap V) \pitchfork (S^{n-1} \cap W)\).