$8-1. \quad \forall p = (x,y,s,t) \in F^{-1}(0,1),
\begin{align*}
D F_p &= \begin{pmatrix} 2x & 1 & 0 & 0 \\
2y & 0 & y+1 & 2s & 2t 
\end{pmatrix} \\
D F_p \text{ does not have rank 2 only if } y = s = t = 0.
\end{align*}

But in this case $F^{-1}(x,0,0,0) = (x^2, x^4) \neq (0,1)$, so $(0,1)$ is a regular value of $F$.

$(x,y,s,t) \in F^{-1}(0,1)$ if and only if $x^2 + y = 0$ and $x^2 + y^2 + s^2 + t^2 + y = 1$, which implies that $y^2 + s^2 + t^2 = 1$ for $-1 \leq y \leq 0$.

Define $\Phi: F^{-1}(0,1) \to S^2$
\[ (x,y,s,t) \mapsto (\text{sgn}(x),y,s,t) \]

Then it's easy to check $\Phi$ is a diffeomorphism.

$8-2. \quad \forall p = (x,y), \quad D F_p = \begin{pmatrix} 3x^2 + y & x + 3y^2 \end{pmatrix}
\]

$D F_p = (0,0)$ only if $3x^2 + y = 0$ and $x + 3y^2 = 0$. Equivalently,
\[ x = y = 0 \quad \text{or} \quad x = y = -\frac{1}{3} \]
\[ F(0,0) = 0, \quad F(-\frac{1}{3}, -\frac{1}{3}) = \frac{1}{27}. \]
So, $\forall c \in \mathbb{R}, c \neq 0$, or $\frac{1}{27}$, $F^{-1}(c)$ is an embedded submanifold of $\mathbb{R}^2$.

$8-1b. \quad \text{Let } \dim M = m, \dim N = n, \quad k = \dim N - \dim S. \quad \forall p \in F^{-1}(S),\]
there exists a neighborhood $U$ of $\Phi(p)$ and a local defining function $\Psi: U \to \mathbb{R}^k$ such that $S \cap U = \Psi^{-1}(0)$.

By Proposition 8.12, it suffices to show that $0$ is a regular value of $\Psi \circ \Phi$, for then $\Psi \circ \Phi$ is a local defining function.
for $S$. Let $Z \in T_0 \mathbb{R}^k$. We have to find $X \in T_0 M$ such that $(\Psi \circ \Phi)^*_x X = Z$, where $P \in (\Psi \circ \Phi)^{-1}(0)$.

To this end, note that $0$ is a regular value of $\Psi$.

So we can find $Y \in T_{\Phi(p)} N$ such that $\Psi^*_x Y = Z$.

By the transversality assumption, we can find $Y_0 \in T_{\Phi(p)} S$ and $X \in T_p M$ such that $Y = Y_0 + \Phi^*_x X$. But $\Psi^*_x Y_0 = 0$ since $\Psi^*_x$ is surjective with kernel $T_{\Phi(p)} S$.

We have $(\Psi \circ \Phi^*_x) X = \Psi^*_x \Phi^*_x X = \Psi^*_x Y = Z$.

(See §1.5 of the book "Differential Topology" by Guillemin and Pollack for more information on transversality.)

§18. Let $m = \dim M$, $n = \dim C$, $k = m - n$. Since $\Phi$ is a global defining function, $C = \Phi^{-1}(0)$ for some $a \in \mathbb{R}^k$ and $a$ is a regular value of $\Phi$. By Proposition 8.12, $f|_C$ is smooth. Let $(U, \Psi)$ be a smooth chart for $C$ with $p \in U$, $\Psi(p) = 0$. Put $f = f \circ \Psi^{-1}$. Then $f$ attains its local maximum or minimum at $0$. So,

$0 = \left. df \right|_0 = (\Psi^{-1})^* df \big|_{\Psi^{-1}(0)}$. Since $(\Psi^{-1})^*$ is an isomorphism, $\left. df \right|_p = 0$ on $T_p C$. On the other hand, $\Phi(C) = A$ implies that $\Phi^*_i$ is constant on $C$. So $\left. d\Phi^*_i \right|_p = 0$ on $T_p C$ for $i = 1, \ldots, k$. Since $p$ is a regular point, $\left< d\Phi^*_i \right>_{i=1}^k$ are linearly independent.
Since $T_p M = T_p C \oplus (T_p C)^\perp$ where $(T_p C)^\perp$ is the orthogonal complement of $T_p C$ in $T_p M$, $dim(T_p C)^\perp = dim(T_p M) - dim(T_p C) = m - n = k$. But $(T_p C)^\perp = \{ \varphi \in (T_p M)^* : \varphi(x) = 0 \text{ for all } x \in T_p C \}$, and we have shown $d f|_p \in (T_p C)^\perp$, $\langle d \xi^i|_p \rangle_{i=1}^k \subset (T_p C)^\perp$, it follows $(T_p C)^\perp = \text{span} \langle d \xi^i|_p \rangle_{i=1}^k$, so there exists $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $d f|_p = \lambda_1 d \xi^1|_p + \ldots + \lambda_k d \xi^k|_p$. 