6-4. \( df = 2x \, dx + 2y \, dy + 2z \, dz \).

\[ F^* df = 2(X \circ F) \, d(X \circ F) + 2(Y \circ F) \, d(Y \circ F) + 2(Z \circ F) \, d(Z \circ F) \]

\[
= \frac{4u}{u^2 + v^2 + 1} \left( -\frac{2u^2 + 2v^2 + 2}{(u^2 + v^2 + 1)^2} \right) \, du - \frac{4uv}{(u^2 + v^2 + 1)^2} \, dv \\
+ \frac{4v}{u^2 + v^2 + 1} \left( \frac{-4uv}{(u^2 + v^2 + 1)^2} \right) \, du + \frac{2u^2 - 2v^2 + 2}{(u^2 + v^2 + 1)^2} \, dv \\
+ \frac{2u^2 + 2v^2 - 2}{u^2 + v^2 + 1} \left( \frac{4u}{(u^2 + v^2 + 1)^2} \right) \, du + \frac{4v}{(u^2 + v^2 + 1)^2} \, dv \\
= 0 \\
(f \circ F) (u, v) = f \left( \frac{u}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right) \\
= \frac{4u^2 + 4v^2 + (u^2 + v^2 - 1)^2}{(u^2 + v^2 + 1)^2} = 1 \\
\text{So } F^* df = df \circ F.

6-5. (a) \( df = \frac{y^2 - x^2}{(x^2 + y^2)^2} \, dx + \frac{-2xy}{(x^2 + y^2)^2} \, dy \)

\( df (x, y) = 0 \) \iff \( x^2 = y^2 \) and \( xy = 0 \)

\( \text{i.e. } x = y = 0 \). But \( (x, y) \notin M \), so \( df (x, y) \neq 0 \)

for \( (x, y) \in M \).

(b) \( f(r, \theta) = \frac{\rho \cos \theta}{r^2} = \frac{\cos \theta}{r} \)

\[ df = \frac{\cos \theta \, r^2 - 2r^2 \cos \theta \, dr - \sin \theta \, d\theta}{r^4} = -\frac{\cos \theta}{r^2} \, dr - \frac{\sin \theta}{r} \, d\theta \]

\( df = 0 \) \iff \( \cos \theta = \sin \theta = 0 \). So \( df \) is never zero.
Let \((s^1 \setminus \mathbb{S}^1, \sigma), (s^2 \setminus \mathbb{S}^1, \tilde{\sigma})\) be the stereographic coordinates, where \(\sigma(u, v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)\), \(\tilde{\sigma}^{-1}(u, v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1}\right)\).

So, \(f(u, v) = \left\{ \begin{array}{ll} \frac{u^2+v^2-1}{u^2+v^2+1} & \text{on } s^1 \setminus \mathbb{S}^1 \\ \frac{1-u^2-v^2}{u^2+v^2+1} & \text{on } s^2 \setminus s^1 \end{array} \right. \)

\(df = \frac{\partial f}{\partial u} \, du + \frac{\partial f}{\partial v} \, dv\)

\(= \left\{ \begin{array}{ll} \frac{4u}{(u^2+v^2+1)^2} \, du + \frac{4v}{(u^2+v^2+1)^2} \, dv & \text{on } s^2 \setminus \mathbb{S}^1 \\ -\frac{4u}{(u^2+v^2+1)^2} \, du + \frac{-4v}{(u^2+v^2+1)^2} \, dv & \text{on } s^2 \setminus s^1 \end{array} \right. \)

so \(df_p = 0 \iff u = v = 0 \)

\(\iff p \) is the south pole or the north pole.

\(f(x) = \sum_{i=1}^{n} x_i^2\)

\(df = \sum_{i=1}^{n} 2x_i \, dx_i\)

\(df_p = 0 \iff x_i = 0 \text{ for } i = 1, \ldots, n\)

\(\iff p = 0\).

6-6. (a) Expanding \(\det X\) by minors along the \(i\)th column,

\(\det X = \sum_{j=1}^{n} (-1)^{i+j} \, x_i^j \, \det M_i^j\) where \(M_i^j\) is the minor of \(X\) without the \(i\)th row and \(j\)th column. By Cramer's rule, \((X^{-1})_i^j = \frac{1}{\det X} \cdot (-1)^{i+j} \, \det M_i^j\).

So \(\frac{\partial}{\partial x_i^j} (\det X) = (-1)^{i+j} \, \det M_i^j = (\det X) \, (X^{-1})_i^j\).
(b) \[ d \left( \det X \right) \mid \partial B = \left( \sum_{i,j} \frac{\partial}{\partial x_i^j} \left( \det X \right) \partial x_i \right) B \]

\[ = \sum_{i,j} \left( \det X \right) \left( X^{-1} \right)_j^i \partial x_i \mid \partial B \quad (\text{by (a)}) \]

\[ = \det X \sum_{i,j} \left( X^{-1} \right)_j^i B_j^i = \left( \det X \right) \text{tr} \left( X^{-1} B \right) \]

6-9. Suppose \( \gamma \) is smooth on \((a_i, b_i)\) for \([a_i, b_i] = \bigcup_{i=1}^{n} [a_i, b_i] \). Then

\[ \int_{F \circ \gamma} x \overset{\gamma \star}{\rightarrow} \nu = \sum_{i=1}^{n} \int_{[a_i, b_i]} (F \circ \gamma)^* w = \sum_{i=1}^{n} \int_{[a_i, b_i]} \gamma^* F^* w \]

\[ = \int_{\gamma} F^* w. \]

Here we used the fact that \( (F \circ \gamma)^* = \gamma^* \circ F^* \)

which follows from Proposition 6.2 and \( F^* = (F \circ \gamma)^* \).

\( (F \circ \gamma)^* = F^* \circ \gamma. \)

6-10. (a) Parametrize the line \((0, 0, 0)\) to \((1, 1, 1)\)

by \( \gamma(t) = (t, t, t), \quad \gamma : [0, 1] \rightarrow \mathbb{R}^3 \). Then

\[ \int_{\gamma} w = \int_0^1 \left( -4t \frac{t}{(t^2 + 1)^2} + \frac{2t}{t^2 + 1} + \frac{2t}{t^2 + 1} \right) dt \]

\[ = 2 \ln 2 - 1. \]

\[ \int_{\gamma} \eta = \int_0^1 \left( -4t^2 \frac{t}{(t^2 + 1)^2} + \frac{2t}{t^2 + 1} + \frac{2t}{t^2 + 1} \right) dt \]

\[ = \ln 2 + 1. \]

(b) \( w \) is not closed (thus not exact) since

\[ \frac{3}{3x^2} \left( -4 \frac{2x}{(x^2 + 1)^2} \right) = -4 \frac{2x}{(x^2 + 1)^2} \neq \frac{3}{3x^2} \left( \frac{2x}{x^2 + 1} \right) = \frac{2 - 2x^2}{(x^2 + 1)^2} \]

\( \eta \) is exact since \( df = \eta \) for \( f = \frac{2x}{x^2 + 1} + \ln (y^2 + 1) \)
(c). \( \int \eta = f(1,1,1) - f(0,0,0) = 1 + \ln 2 \).

6-13. (a)

Let \( \omega \) be an exact covector field on \( M \) and \( \omega = df \)
for some smooth function \( f: M \to \mathbb{R} \). \( M \) is compact, so there exist \( p \neq q \in M \) s.t. \( f \) attains its absolute maximum and minimum at \( p \) and \( q \), respectively. Consider the coordinate representation of \( f \) in a smooth chart \((\phi, U)\) containing \( p \), \( U = \phi^{-1} \). Then
\[
\frac{\partial f}{\partial x_i}(\phi(p)) = 0 \quad \text{for} \quad i = 1, \ldots, n.
\]
So
\[
df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\phi(p)) \, dx_i(\phi^{-1})_p = 0.
\]

Similarly \( df_q = 0 \).

(b). Note that the change of variable formulas for tangent vectors and tangent covectors are similar. We can construct a smooth covector on \( S^2 \) that vanishes at exactly one point following the solution of Problem 4-6. Indeed, let \( \{S^1 \times \{y\}, \delta_j \} \) and \( \{S^1 \times \{y\}, \delta_j \} \) be the stereographic charts. Then we can define
\[
\omega = \int \left( (u^2 - u^2) \, du - 2uv \, dv \right) \text{ on } S^1 \times \{y\}, \delta_j.
\]

One can check \( \omega \) is smooth following verbatim Problem 4-6. \( \omega \) only vanishes at \( 0 \).