$W \subset M$ containing p and a smooth map $\widetilde{F}: W \to N$ whose restriction to $W \cap A$ agrees with F.

Lemma 2.27 (Extension Lemma). *Let M be a smooth manifold, let $A \subset M$ be a closed subset, and let $f: A \to \mathbb{R}^k$ be a smooth function. For any open set U containing A, there exists a smooth function $\widetilde{f}: M \to \mathbb{R}^k$ such that $\widetilde{f}|_A = f$ and $\operatorname{supp} \widetilde{f} \subset U$.*

Proof. For each $p \in A$, choose a neighborhood W_p of p and a smooth function $\widetilde{f}_p: W_p \to \mathbb{R}^k$ that agrees with f on $W_p \cap A$. Replacing W_p by $W_p \cap U$, we may assume that $W_p \subset U$. The collection of sets $\{W_p : p \in A\} \cup \{M \setminus A\}$ is an open cover of M. Let $\{\psi_p : p \in A\} \cup \{\psi_0\}$ be a smooth partition of unity subordinate to this cover, with $\operatorname{supp} \psi_p \subset W_p$ and $\operatorname{supp} \psi_0 \subset M \setminus A$.

For each $p \in A$, the product $\psi_p \widetilde{f}_p$ is smooth on W_p, and has a smooth extension to all of M if we interpret it to be zero on $M \setminus \operatorname{supp} \psi_p$. (The extended function is smooth because the two definitions agree on the open set $W_p \setminus \operatorname{supp} \psi_p$ where they overlap.) Thus we can define $\widetilde{f}: M \to \mathbb{R}^k$ by

$$\widetilde{f}(x) = \sum_{p \in A} \psi_p(x) \widetilde{f}_p(x).$$

Because the collection of supports $\{\operatorname{supp} \psi_p\}$ is locally finite, this sum actually has only a finite number of nonzero terms in a neighborhood of any point of M, and therefore defines a smooth function. If $x \in A$, then $\widetilde{f}_p(x) = f(x)$ for each p and $\psi_0(x) = 0$, and thus

$$\widetilde{f}(x) = \sum_{p \in A} \psi_p(x) f(x) = \left(\psi_0(x) + \sum_{p \in A} \psi_p(x) \right) f(x) = f(x),$$

so $\widetilde{f}$ is indeed an extension of f. Finally, suppose $x \in \operatorname{supp} \widetilde{f}$. Then x has a neighborhood on which at most finitely many of the functions ψ_p are nonzero, and x must be in $\operatorname{supp} \psi_p$ for at least one $p \in A$, which implies that $x \in W_p \subset U$. □

The extension lemma, by the way, illustrates an essential difference between smooth manifolds and real-analytic manifolds. The analogue of the extension lemma for real-analytic functions on real-analytic manifolds is decidedly false, because a real-analytic function that is defined on a connected domain and vanishes on an open set must be identically zero.

As our final application of partitions of unity, we will construct a special kind of smooth function. If M is a topological space, an *exhaustion function* for M is a continuous function $f: M \to \mathbb{R}$ with the property that the set $M_c = \{x \in M : f(x) \le c\}$ is compact for each $c \in \mathbb{R}$. The name comes from the fact that the compact sets M_c exhaust M as c increases to positive infinity. For example, the functions $f: \mathbb{R}^n \to \mathbb{R}$ and $g: \mathbb{B}^n \to \mathbb{R}$ given by

$$f(x) = |x|, \qquad g(x) = \frac{1}{1 - |x|^2},$$

are exhaustion functions. Of course, if M is compact, any continuous real-valued function on M is an exhaustion function, so such functions are interesting only for noncompact manifolds.

Proposition 2.28 (Existence of Exhaustion Functions). *Every smooth manifold admits a smooth positive exhaustion function.*

Proof. Let M be a smooth manifold, let $\{V_j\}_{j=1}^\infty$ be any countable open cover of M by precompact open sets, and let $\{\psi_j\}$ be a smooth partition of unity subordinate to this cover. Define $f \in C^\infty(M)$ by

$$f(p) = \sum_{j=1}^\infty j \psi_j(p).$$

Then f is smooth because only finitely many terms are nonzero in a neighborhood of any point, and positive because $f(p) \ge \sum_j \psi_j(p) = 1$. For any positive integer N, if $p \notin \bigcup_{j=1}^N \overline{V}_j$, then $\psi_j(p) = 0$ for $1 \le j \le N$, so

$$f(p) = \sum_{j=N+1}^\infty j \psi_j(p) > \sum_{j=N+1}^\infty N \psi_j(p) = N \sum_{j=1}^\infty \psi_j(p) = N.$$

Equivalently, if $f(p) \le N$, then $p \in \bigcup_{j=1}^N \overline{V}_j$. Thus for any $c \le N$, M_c is a closed subset of the compact set $\bigcup_{j=1}^N \overline{V}_j$ and is therefore compact. □

Problems

2-1. Compute the coordinate representation for each of the following maps in stereographic coordinates (see Problem 1-5), and use this to prove that each map is smooth.

(a) For each $n \in \mathbb{Z}$, the *nth power map* $p_n: \mathbb{S}^1 \to \mathbb{S}^1$ is given in complex notation by $p_n(z) = z^n$.
(b) $\alpha: \mathbb{S}^n \to \mathbb{S}^n$ is the *antipodal map* $\alpha(x) = -x$.
(c) $F: \mathbb{S}^3 \to \mathbb{S}^2$ is given by $F(z, w) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w})$, where we think of $\mathbb{S}^3$ as the subset $\{(w, z) : |w|^2 + |z|^2 = 1\}$ of $\mathbb{C}^2$.

2-2. Show that the inclusion map $\overline{\mathbb{B}^n} \hookrightarrow \mathbb{R}^n$ is smooth when $\overline{\mathbb{B}^n}$ is regarded as a smooth manifold with boundary.

2-3. Let $\mathbf{R}$ denote the real line with its standard smooth structure, and let $\widetilde{\mathbf{R}}$ denote the same topological manifold with the smooth structure defined in Example 1.14. Let $f: \mathbb{R} \to \mathbb{R}$ be any function. Determine necessary and sufficient conditions on f so that it will be:

(a) a smooth map from $\mathbb{R}$ to $\widetilde{\mathbb{R}}$;
(b) a smooth map from $\widetilde{\mathbb{R}}$ to $\mathbb{R}$.

2-4. Let $P: \mathbb{R}^{n+1} \smallsetminus \{0\} \to \mathbb{R}^{k+1} \smallsetminus \{0\}$ be a smooth map, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \smallsetminus \{0\}$ and $x \in \mathbb{R}^{n+1} \smallsetminus \{0\}$. (Such a map is said to be *homogeneous of degree d*.) Show that the map $\widetilde{P}: \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\widetilde{P}[x] = [P(x)]$ is well-defined and smooth.

2-5. Let M be a nonempty smooth manifold of dimension $n \geq 1$. Show that $C^\infty(M)$ is infinite-dimensional.

2-6. For any topological space M, let $C(M)$ denote the algebra of continuous functions $f: M \to \mathbb{R}$. If $F: M \to N$ is a continuous map, define $F^*: C(N) \to C(M)$ by $F^*(f) = f \circ F$.

(a) Show that F^* is a linear map.
(b) If M and N are smooth manifolds, show that F is smooth if and only if $F^*(C^\infty(N)) \subset C^\infty(M)$.
(c) If $F: M \to N$ is a homeomorphism between smooth manifolds, show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

[Remark: This result shows that in a certain sense, the entire smooth structure of M is encoded in the space $C^\infty(M)$. In fact, some authors *define* a smooth structure on a topological manifold M to be a subalgebra of $C(M)$ with certain properties.]

2-7. Let M be a connected smooth manifold, and let $\pi: \widetilde{M} \to M$ be a topological covering map. Show that there is only one smooth structure on $\widetilde{M}$ such that π is a smooth covering map (see Proposition 2.12). [Hint: Use the existence of smooth local sections.]

2-8. Show that the map $\varepsilon^n: \mathbb{R}^n \to \mathbb{T}^n$ defined in Example 2.8(d) is a smooth covering map.

2-9. Show that the map $p: \mathbb{S}^n \to \mathbb{RP}^n$ defined in Example 2.5(d) is a smooth covering map.

2-10. Let $\mathbb{CP}^n$ denote n-dimensional complex projective space, as defined in Problem 1-7.

(a) Show that the quotient map $\pi: \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{CP}^n$ is smooth.
(b) Show that $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$.

2-11. Let G be a connected Lie group, and let $U \subset G$ be any neighborhood of the identity. Show that every element of G can be written as a finite product of elements of U. In particular, U generates G. (A subset U of a group G is said to *generate* G if every element of G can be written as a finite product of elements of U and their inverses.)

2-12. Let G be a Lie group, and let G_0 denote the connected component of G containing the identity (called the *identity component* of G).

(a) Show that G_0 is the only connected open subgroup of G.
(b) Show that each connected component of G is diffeomorphic to G_0.

2-13. Let G be a connected Lie group. Show that the universal covering group $\widetilde{G}$ constructed in Theorem 2.13 is unique in the following sense: If G' is any other simply connected Lie group that admits a smooth covering map $\pi': G' \to G$ that is also a Lie group homomorphism, then there exists a Lie group isomorphism $\Phi: \widetilde{G} \to G'$ such that $\pi' \circ \Phi = \pi$.

2-14. Let M be a topological manifold, and let $\mathcal{U}$ be a cover of M by precompact open sets. Show that $\mathcal{U}$ is locally finite if and only if each set in $\mathcal{U}$ intersects only finitely many other sets in $\mathcal{U}$. Give counterexamples to show that the conclusion is false if either precompactness or openness is omitted from the hypotheses.

2-15. Suppose M is a locally Euclidean Hausdorff space. Show that M is second countable if and only if it is paracompact and has countably many connected components. [Hint: If M is paracompact, show that each component of M has a locally finite cover by precompact coordinate balls, and extract from this a countable subcover.]

2-16. Suppose M is a topological space with the property that for every open cover $\mathcal{X}$ of M, there exists a partition of unity subordinate to $\mathcal{X}$. Show that M is paracompact.

2-17. Show that the assumption that A is closed is necessary in the extension lemma (Lemma 2.27), by giving an example of a smooth real-valued function on a nonclosed subset of a smooth manifold that admits no smooth extension to the whole manifold.

2-18. Let M be a smooth manifold, let $B \subset M$ be a closed subset, and let $\delta: M \to \mathbb{R}$ be a positive continuous function.

(a) Using a partition of unity, show that there is a smooth function $\widetilde{\delta}: M \to \mathbb{R}$ such that $0 < \widetilde{\delta}(x) < \delta(x)$ for all $x \in M$.
(b) Show that there is a continuous function $\psi: M \to \mathbb{R}$ that is smooth and positive on $M \smallsetminus B$, zero on B, and satisfies $\psi(x) < \delta(x)$ everywhere. [Hint: Consider $1/(1+f)$, where $f: M \smallsetminus B \to \mathbb{R}$ is a positive exhaustion function.]