Lemma 2.27 (Extension Lemma). Let $M$ be a smooth manifold, let $A \subset M$ be a closed subset, and let $f: A \to \mathbb{R}^k$ be a smooth function. For any open set $U$ containing $A$, there exists a smooth function $\tilde{f}: M \to \mathbb{R}^k$ such that $f|_A = f$ and $\tilde{f} \subset U$.

Proof. For each $p \in A$, choose a neighborhood $W_p$ of $p$ and a smooth function $f_p: W_p \to \mathbb{R}^k$ that agrees with $f$ on $W_p \cap A$. Replacing $W_p$ by $W_p \cap U$, we may assume that $W_p \subset U$. The collection of sets $\{W_p : p \in A\} \cup \{M \setminus A\}$ is an open cover of $M$. Let $\{\psi_p : p \in A\} \cup \{\psi_0\}$ be a smooth partition of unity subordinate to this cover, with $\text{supp} \psi_p \subset W_p$ and $\text{supp} \psi_0 \subset M \setminus A$.

For each $p \in A$, the product $\psi_p f_p$ is smooth on $W_p$, and has a smooth extension to all of $M$ if we interpret it to be zero on $M \setminus \text{supp} \psi_p$. (The extended function is smooth because the two definitions agree on the open set $W_p \cap \text{supp} \psi_p$ where they overlap.) Thus we can define $\tilde{f}: M \to \mathbb{R}^k$ by

$$\tilde{f}(x) = \sum_{p \in A} \psi_p(x) f_p(x).$$

Because the collection of supports $\{\text{supp} \psi_p\}$ is locally finite, this sum actually has only a finite number of nonzero terms in a neighborhood of any point of $M$, and therefore defines a smooth function. If $x \in A$, then $\tilde{f}(x) = f(x)$ for each $p$ and $\psi_0(x) = 0$, and thus

$$\tilde{f}(x) = \sum_{p \in A} \psi_p(x) f(x) = \left(\psi_0(x) + \sum_{p \in A} \psi_p(x)\right) f(x) = f(x),$$

so $\tilde{f}$ is indeed an extension of $f$. Finally, suppose $x \in \text{supp} \tilde{f}$. Then $x$ is a neighborhood on which at most finitely many of the functions $\psi_p$ are nonzero, and $x$ must be in $\text{supp} \psi_p$ for at least one $p \in A$, which implies that $x \in W_p \subset U$.

The extension lemma, by the way, illustrates an essential difference between smooth manifolds and real-analytic manifolds. The analogue of the extension lemma for real-analytic functions on real-analytic manifolds is decidedly false, because a real-analytic function that is defined on a connected domain and vanishes on an open set must be identically zero.

As our final application of partitions of unity, we will construct a special kind of smooth function. If $M$ is a topological space, an exhaustion function for $M$ is a continuous function $f: M \to \mathbb{R}$ with the property that the sets $M_c = \{x \in M : f(x) \leq c\}$ is compact for each $c \in \mathbb{R}$. The name comes from the fact that the compact sets $M_c$ exhaust $M$ as $c$ increases to positive infinity. For example, the functions $f: \mathbb{R}^n \to \mathbb{R}$ and $g: \mathbb{B}^n \to \mathbb{R}$ given by

$$f(x) = |x|,$$

$$g(x) = \frac{1}{1 - |x|^2},$$

are exhaustion functions. Of course, if $M$ is compact, any continuous real-valued function on $M$ is an exhaustion function, so such functions are interesting only for noncompact manifolds.

Proposition 2.28 (Existence of Exhaustion Functions). Every smooth manifold admits a smooth positive exhaustion function.

Proof. Let $M$ be a smooth manifold, let $\{V_j\}_{j=1}^\infty$ be any countable open cover of $M$ by precompact open sets, and let $\{\psi_j\}$ be a smooth partition of unity subordinate to this cover. Define $f \in C^\infty(M)$ by

$$f(p) = \sum_{j=1}^\infty j \psi_j(p).$$

Then $f$ is smooth because only finitely many terms are nonzero in a neighborhood of any point, and positive because $f(p) \geq \sum_{j} \psi_j(p) = 1$. For any positive integer $N$, if $p \not\in \bigcup_{j=1}^N \overline{V}_j$, then $\psi_j(p) = 0$ for all $1 \leq j \leq N$, so

$$f(p) = \sum_{j=N+1}^\infty j \psi_j(p) > \sum_{j=N+1}^\infty N \psi_j(p) = N \sum_{j=1}^\infty \psi_j(p) = N.$$

Equivalently, if $f(p) \leq N$, then $p \in \bigcup_{j=1}^N \overline{V}_j$. Thus for any $c \leq N$, $M_c$ is a closed subset of the compact set $\bigcup_{j=1}^N \overline{V}_j$ and is therefore compact. \(\square\)

Problems

2.1. Compute the coordinate representation for each of the following maps in stereographic coordinates (see Problem 1-5), and use this to prove that each map is smooth.

(a) For each $n \in \mathbb{Z}$, the $n$th power map $p_n: S^1 \to S^1$ is given in complex notation by $p_n(z) = z^n$.

(b) $\alpha: S^n \to S^n$ is the antipodal map $\alpha(z) = -z$.

(c) $F: S^3 \to S^2$ is given by $F(x, w) = (x\overline{w} + w\overline{x}, i(x\overline{w} - w\overline{x}), z\overline{x} - w\overline{z})$, where we think of $S^3$ as the subset $\{(w, z) : |w|^2 + |z|^2 = 1\}$ of $\mathbb{C}^2$.

2.2. Show that the inclusion map $\mathbb{B}^n \hookrightarrow \mathbb{R}^n$ is smooth when $\mathbb{B}^n$ is regarded as a smooth manifold with boundary.

2.3. Let $\mathbb{R}$ denote the real line with its standard smooth structure, and let $\mathbb{R}$ denote the same topological manifold with the smooth structure
defined in Example 1.14. Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be any function. Determine necessary and sufficient conditions on \( f \) so that it will be:

(a) a smooth map from \( \mathbb{R} \) to \( \mathbb{R} \);
(b) a smooth map from \( \mathbb{R} \) to \( \mathbb{R} \).

2-4. Let \( P: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}^{k+1} \setminus \{0\} \) be a smooth map, and suppose that for some \( d \in \mathbb{Z} \), \( P(\lambda x) = \lambda^d P(x) \) for all \( \lambda \in \mathbb{R} \setminus \{0\} \) and \( x \in \mathbb{R}^{n+1} \setminus \{0\} \). (Such a map is said to be homogeneous of degree \( d \).) Show that the map \( \tilde{P}: \mathbb{RP}^n \rightarrow \mathbb{RP}^k \) defined by \( \tilde{P}[x] = [P(x)] \) is well-defined and smooth.

2-5. Let \( M \) be a nonempty smooth manifold of dimension \( n \geq 1 \). Show that \( C^\infty(M) \) is infinite-dimensional.

2-6. For any topological space \( M \), let \( C(M) \) denote the algebra of continuous functions \( f: M \rightarrow \mathbb{R} \). If \( f: M \rightarrow N \) is a continuous map, define \( f^*: C(N) \rightarrow C(M) \) by \( f^*(f) = f \circ f \).

(a) Show that \( f^* \) is a linear map.
(b) If \( M \) and \( N \) are smooth manifolds, show that \( f \) is smooth if and only if \( f^*(C^\infty(N)) \subseteq C^\infty(M) \).
(c) If \( f: M \rightarrow N \) is a homeomorphism between smooth manifolds, show that it is a diffeomorphism if and only if \( f^* \) restricts to an isomorphism from \( C^\infty(N) \) to \( C^\infty(M) \).

[Remark: This result shows that in a certain sense, the entire smooth structure of \( M \) is encoded in the space \( C^\infty(M) \). In fact, some authors define a smooth structure on a topological manifold \( M \) to be a subalgebra of \( C(M) \) with certain properties.]

2-7. Let \( M \) be a connected smooth manifold, and let \( \pi: \tilde{M} \rightarrow M \) be a topological covering map. Show that there is only one smooth structure on \( M \) such that \( \pi \) is a smooth covering map (see Proposition 2.12). [Hint: Use the existence of smooth local sections.]

2-8. Show that the map \( c^n: \mathbb{R}^n \rightarrow \mathbb{T}^n \) defined in Example 2.8(d) is a smooth covering map.

2-9. Show that the map \( p: S^n \rightarrow \mathbb{R}P^n \) defined in Example 2.5(d) is a smooth covering map.

2-10. Let \( \mathbb{CP}^n \) denote \( n \)-dimensional complex projective space, as defined in Problem 1-7.

(a) Show that the quotient map \( \pi: \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{CP}^n \) is smooth.
(b) Show that \( \mathbb{CP}^1 \) is diffeomorphic to \( S^2 \).

2-11. Let \( G \) be a connected Lie group, and let \( U \subset G \) be any neighborhood of the identity. Show that every element of \( G \) can be written as a finite product of elements of \( U \). In particular, \( U \) generates \( G \). (A subset \( U \) of a group \( G \) is said to generate \( G \) if every element of \( G \) can be written as a finite product of elements of \( U \) and their inverses.)

2-12. Let \( G \) be a Lie group, and let \( G_0 \) denote the connected component of \( G \) containing the identity (called the identity component of \( G \)).

(a) Show that \( G_0 \) is the only connected open subgroup of \( G \).
(b) Show that each connected component of \( G \) is diffeomorphic to \( G_0 \).

2-13. Let \( G \) be a connected Lie group. Show that the universal covering group \( \tilde{G} \) constructed in Theorem 2.13 is unique in the following sense: If \( G' \) is any other simply connected Lie group that admits a smooth covering map \( \pi': G' \rightarrow G \) that is also a Lie group homomorphism, then there exists a Lie group isomorphism \( \Phi: \tilde{G} \rightarrow G' \) such that \( \pi' \circ \Phi = \pi \).

2-14. Let \( M \) be a topological manifold, and let \( \mathcal{U} \) be a cover of \( M \) by precompact open sets. Show that \( \mathcal{U} \) is locally finite if and only if each set in \( \mathcal{U} \) intersects only finitely many other sets in \( \mathcal{U} \). Give counterexamples to show that the conclusion is false if either precompactness or openness is omitted from the hypotheses.

2-15. Suppose \( M \) is a locally Euclidean Hausdorff space. Show that \( M \) is second countable if and only if it is paracompact and has countably many connected components. [Hint: If \( M \) is paracompact, show that each component of \( M \) has a locally finite cover by precompact coordinate balls, and extract from this a countable subcover.]

2-16. Suppose \( M \) is a topological space with the property that for every open cover \( \mathcal{X} \) of \( M \), there exists a partition of unity subordinate to \( \mathcal{X} \). Show that \( M \) is paracompact.

2-17. Show that the assumption that \( N \) is closed is necessary in the extension lemma (Lemma 2.27), by giving an example of a smooth real-valued function on a nonclosed subset of a smooth manifold that admits no smooth extension to the whole manifold.

2-18. Let \( M \) be a smooth manifold, let \( U \subset M \) be a closed subset, and let \( \delta: M \rightarrow \mathbb{R} \) be a positive continuous function.

(a) Using a partition of unity, show that there is a smooth function \( \tilde{\delta}: M \rightarrow \mathbb{R} \) such that \( 0 < \tilde{\delta}(x) < \delta(x) \) for all \( x \in M \).
(b) Show that there is a continuous function \( \psi: M \rightarrow \mathbb{R} \) that is smooth and positive on \( M \setminus B \), zero on \( B \), and satisfies \( \psi(x) < \delta(x) \) everywhere. [Hint: Consider \( 1/(1 + f) \), where \( f: M \setminus B \rightarrow \mathbb{R} \) is a positive exhaustion function.]