1. Problem 1

Since $\dim(V) = m$, we have $\dim(\Lambda^m(V)) = 1$ with generator $\omega_0 = dx_1 \wedge \cdots \wedge dx_m$ in the coordinates on $V$ with respect to a basis $(e_1, \ldots, e_m)$. So it is enough to compute $A^*\omega_0$. Let $A = (a_{ij})$ be the matrix form. We have $A^*dx_i(e_k) = dx_i(Ae_k) = a_{ik}$. So

$$A^*dx_i = \sum_k a_{ik}dx_k$$

If $S_m$ denotes the group of permutations over $m$ letters, then

$$A^*\omega_0 = \left(\sum a_{1k}dx_k\right) \wedge \cdots \wedge \left(\sum a_{mk}dx_k\right)$$

$$= \sum_{\sigma \in S_m} (a_{1\sigma(1)} \cdots a_{m\sigma(m)})dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(m)}$$

$$= \sum_{\sigma \in S_m} (-1)^{sgn(\sigma)}(a_{1\sigma(1)} \cdots a_{m\sigma(m)})dx_1 \wedge \cdots \wedge dx_m$$

$$= \det(A)\omega_0$$

So $A^*\omega = \det(A)\omega$.

Remark 1.1. Suppose $f$ is a diffeomorphism from an open set $U$ in $\mathbb{R}^m$ to some other open set $V$. Then by a similar sort of calculation, you can check that when the standard volume form $\omega = dx_1 \wedge \cdots \wedge dx_m$ is pulled back by $f$, it gets multiplied by the Jacobian of $f$, i.e. if $y = f(x)$

$$df^*\omega(x) = \det(Df(y))\omega$$

This illustrates the main point about differential forms: the change of variables formula gets automatically built in because of the above equation, and so the integral of a differential form is well-defined and independent of the choice of co-ordinates. This allows us to integrate on manifolds with no reference to any embedding in some Euclidean space.

2. Problem 2

When $V$ is 1-dimensional, $\Lambda^2(V) = 0$.

When $V$ is 2-dimensional, $\Lambda^2(V)$ is one dimensional spanned by $dx_1 \wedge dx_2$. So it is obviously decomposable, and hence so is $f(x)dx_1 \wedge dx_2$.

When $V$ is 3-dimensional, $\Lambda^2(V)$ is spanned by the forms $dx_1 \wedge dx_2, dx_2 \wedge dx_3, dx_3 \wedge dx_1$. Suppose at a point $x \in V$, we have a 2-form $\omega = a_{12}dx_1 \wedge dx_2 + a_{23}dx_2 \wedge dx_3 + a_{31}dx_3 \wedge dx_1$. If it were to be decomposable, we should be able to write it as

$$\omega = (b_1dx_1 + b_2dx_2 + b_3dx_3) \wedge (c_1dx_1 + c_2dx_2 + c_3dx_3)$$

$$= (b_1c_2 - b_2c_1)dx_1 \wedge dx_2 + (b_2c_3 - b_3c_2)dx_2 \wedge dx_3 + (b_3c_1 - b_1c_3)dx_3 \wedge dx_1$$
This gives us equations
\[ b_1c_2 - b_2c_1 = a_{12} \]
\[ b_2c_3 - b_3c_2 = a_{23} \]
\[ b_3c_1 - b_1c_3 = a_{31} \]

Check directly that the above system always has solutions.

Consider now the same problem in \( \mathbb{R}^4 \), and let \( \omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \). We claim that this is not decomposable. Suppose on the contrary that it is decomposable. Write it as
\[ \omega = \left( \sum b_i dx_i \right) \wedge \left( \sum c_i dx_i \right) = \alpha_1 \wedge \alpha_2 \]

First, note that all \( b_i \) and \( c_i \) have to be nonzero. For instance, if \( b_1 = 0 \) then to have the term \( dx_1 \wedge dx_2 \) in \( \omega \) we must have \( c_1 \neq 0 \). But then \( \omega \) also contains terms linear in \( dx_1 \wedge dx_3 \) and \( dx_1 \wedge dx_4 \).

Denote the inclusion of \( \mathbb{R}^2 \) in \( \mathbb{R}^4 \) as a subspace spanned by the basis elements \( \{e_i, e_j\} \) by \( V_{ij} \). The pullback of \( \omega \) to \( V_{13} \) by its inclusion map is 0. The pullback of the 1-forms \( \alpha_1 \) and \( \alpha_2 \) to \( V_{13} \) are \( b_1 dx_1 + b_3 dx_3 \) and \( c_1 dx_1 + c_3 dx_3 \). Since the wedge product commutes with pullbacks, the forms \( b_1 dx_1 + b_3 dx_3 \) and \( c_1 dx_1 + c_3 dx_3 \) are linearly dependent. Applying similar logic to other such subspaces we get that the corresponding pullback one forms are linearly dependent. Since all \( b_i \) and \( c_i \) are non-zero, this forces \( \alpha_2 = r \alpha_1 \). But then, \( \alpha_1 \wedge \alpha_2 = 0 \), a contradiction.

3. Problem 6

Suppose \( \omega \) is a 1-form on \( S_1 \) and let
\[ I_\omega = \int_{S_1} \omega \]

Let
\[ I_\nu = \int_{S_1} \nu \]

and set \( c = I_\omega / I_\nu \) which is well-defined since \( I_\nu \neq 0 \). Check that
\[ \int_{S_1} (\omega - c\nu) = 0 \]

By Problem 5, there is a function \( f \) such that \( \omega - c\nu = df \).