(a) Let \((U, \phi)\) and \((V, \psi)\) be 2 charts around \(p\).

Then at \(p\),

\[
g(\psi) \frac{\partial}{\partial \psi} = \sqrt{f(x)} \frac{\partial}{\partial x} \quad \text{at } p
\]

\[
= g(\psi) = f(\psi) \frac{\partial}{\partial x}
\]

(apply \(\frac{\partial}{\partial x}\))

\[
\Rightarrow \frac{\partial f}{\partial \psi} \frac{\partial \psi}{\partial x} = \frac{\partial f}{\partial x} \frac{\partial \psi}{\partial x} + f \cdot \frac{\partial^2 \psi}{\partial x^2}
\]

Evaluate both sides at \(p\), i.e., \(f(p) = 0\),

we have \(\frac{\partial g}{\partial \psi} = \frac{\partial f}{\partial x} \quad \text{at } p\)

\[\text{WLOG}\]

(b) Suppose \(X\) has infinite norm \(\{x_n\} < S\), the \(x_n \to p \in S\).

Choose a coord chart \((U, \phi)\) around \(p\), then on \(U\),

\[X = f \frac{\partial}{\partial x}\]

and \(f(p_n) = 0\) where \(p_n = \phi(x_n)\)

by continuity \(f(0) = 0\).

By hypothesis, \(\frac{\partial f}{\partial x}|_{x=0} \neq 0\),

however \(\frac{\partial f}{\partial x}|_{x=0} = \lim_{p_n \to 0} \frac{f(p_n) - f(0)}{p_n - 0} = 0\). contradiction.
2a) We know that $GL(3, \mathbb{R})$ is an open submanifold of $M(3, \mathbb{R})$ and $O(3, \mathbb{R})$ is a submanifold of $GL(3, \mathbb{R})$.

Consider $F : O(3, \mathbb{R}) \to \mathbb{R}$ given by $F(A) = \det A$ for all $A \in O(3, \mathbb{R})$.

It is clear that $SO(3, \mathbb{R}) = F^{-1}(1)$ and $1$ is a regular value of $F$.

Since $SO(3, \mathbb{R})$ is a submanifold of $O(3, \mathbb{R})$, hence of $M(3, \mathbb{R})$.

2b) Consider $F : \mathbb{R}^6 \to \mathbb{R}^3$ defined by $F(x_1, x_2, x_3, p_1, p_2, p_3) = (x_1^2 + x_2^2 + x_3^2, p_1^2 + p_2^2 + p_3^2, x_1 p_1 + x_2 p_2 + x_3 p_3)$

Let $L \subset \mathbb{R}^3$ be the set of unit vectors tangent to the sphere, then $L = F^{-1}(1, 1, 0)$.

For all $(x_1, x_2, x_3, p_1, p_2, p_3) \in F^{-1}(1, 1, 0)$

$$DF_{(x, p)} = \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2p_1 & 2p_2 & 2p_3 \\ p_1 & p_2 & p_3 & x_1 & x_2 & x_3 \end{pmatrix}$$
Since \(|x| = |p| = 1\), the first two rows are linearly independent:

\[ x = (x_1, x_2, x_3), \quad p = (p_1, p_2, p_3) \]

Since \(x \cdot p = 0\), the third row cannot be in the span of the first 2 rows, hence \(DF_{(x, p)}\) has full rank. This means \((1, 0, 0)\) is a regular value of \(F\), and \(L\) is a smooth submanifold.

\[ \sum_{x=1}^{3} x_1^2 = 1 \quad \text{and} \quad \sum_{x=1}^{3} x_2^2 = 1 \quad \text{iff} \quad \sum_{x=1}^{3} x_i^2 = 1 \quad \text{and} \quad y_1 = y_2 = y_3 = 0 \]

where \(z = x + x'y\).

Define \(F : \mathbb{R}^6 \rightarrow \mathbb{R}^4\) by

\[ F(x_1, x_2, x_3) = F(x_1, y_2, x_3, y_1, y_2, y_3) = (\sum_{x=1}^{3} x_i^2, y_1, y_2, y_3) \]

Note \(F^{-1}(1, 0, 0, 0) = \{ \sum_{x=1}^{3} x_i^2 = 1 \} \cap \{ \sum_{x=1}^{3} x_i^2 = 1 \} \)

\(F(z_1, z_2, z_3) \in F^{-1}(1, 0, 0, 0)\)

\[ DF_{(z_1, z_2, z_3)} = \begin{pmatrix}
2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \]

which has full rank since \(\sum_{x=1}^{3} x_i^2 = 1\).

Hence \(F^{-1}(1, 0, 0, 0)\) is a submanifold.
(a) Say for $\sigma_N: S^3 \setminus N^3 \to \mathbb{R}^3$, $\sigma_N(x_1, x_2, x_3, x_4) = \left( \frac{x_1}{1-x_4}, \frac{x_2}{1-x_3}, \frac{x_3}{1-x_4} \right)$

so if $e^{it} \vec{z}^2 \neq i$, i.e. $y_2 \cos t + x_2 \sin t \neq 1$ where $\vec{z}^2 = x_2 + i y_2$

we have $\sigma_N(y^2(t)) = $

\[
\left( \frac{x_1 \cos t - y_1 \sin t}{1 - (y_2 \cos t + x_2 \sin t)}, \frac{y_1 \cos t + x_1 \sin t}{1 - (y_2 \cos t + x_2 \sin t)}, \frac{x_2 \cos t - y_2 \sin t}{1 - (y_2 \cos t + x_2 \sin t)} \right)
\]

which is smooth as long as $y^2(t) \in S^3 \setminus N^3$

\((=) e^{it} \vec{z}^2 \neq i.

Similarly we can check that $\sigma_S(y^2(t))$ is also smooth. \]

(b) For $y^2(t) = (e^{it} \vec{z}^1, e^{it} \vec{z}^2) \neq (0, i)$, we compute

that $\frac{d}{dt}(\sigma_N(y^2(t))) = \left( \frac{x_1 x_2 + y_1 y_2 - (x_1 \sin t + y_1 \cos t)}{(1 - y_2 \cos t - x_2 \sin t)^2}, \frac{y_1 x_2 - x_1 y_2 + (x_1 \cos t - y_1 \sin t)}{(1 - y_2 \cos t - x_2 \sin t)^2}, \frac{y_1^2 + y_2^2 - (x_2 \sin t + y_2 \cos t)}{(1 - y_2 \cos t - x_2 \sin t)^2} \right)$

If $\frac{d}{dt}(\sigma_N(y^2(t))) \neq 0$, then we compute that:

\[(x_1 x_2 + y_1 y_2)^2 + (y_1 x_2 - x_1 y_2)^2 = (x_1 \sin t + y_1 \cos t)^2 + (x_1 \cos t - y_1 \sin t)^2.\]
\[ (x_1^2 + y_1^2)(x_2^2 + y_2^2 - 1) = 0. \]

If \[ x_1^2 + y_1^2 = 0, \] then \[ |z_1|^2 + |z_2|^2 = 1 \] \[ \Rightarrow \] \[ x_2^2 + y_2^2 = 1 \] \[ \text{or} \quad |e^{it} z_2|^2 = 1. \]

\[ = \] \[ x_2 \sin t + y_2 \cos t = 1, \] \[ \text{or} \quad \text{Im} (e^{it} z_2) = 1 \]

in view of the 3rd coordinate of \[ \frac{d}{dt} (\sigma_3 (\gamma_3(t))). \]

Now \[ |e^{it} z_2|^2 = 1 \] and \[ \text{Im} (e^{it} z_2) = 1 \]

\[ \Rightarrow e^{it} z_2 = i. \]

\[ \text{contradicts} \quad \gamma_3(t) \in S^3 \setminus \{N\} \]

Similarly we can prove \[ \frac{d}{dt} (\sigma_3 (\gamma_3(t))) \neq 0 \]

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7-2 Since \( F(x, y, z) = F(-x, -y, -z) \), \( F \) descends to a map from \( \mathbb{RP}^2 \) to \( \mathbb{R}^4 \), that map is again denoted by \( F \).

\( DF \) is clearly injective by looking at its Jacobian matrix.

Suppose \( F(x_1, y_1, z_1) = F(x_2, y_2, z_2) \) and \( \sum x_i^2 + y_i^2 + z_i^2 = 1. \)

Then \( x_1^2 - y_1^2 = x_2^2 - y_2^2 \) and \( x_1 y_1 = x_2 y_2 \)

\[ \Rightarrow \quad \pm (x_1 + i y_1) = x_2 + i y_2 \]

By looking at the other two equations from \( F(x_1, y_1, z_1) = F(x_2, y_2, z_2) \)

we see that \( \pm z_1 = z_2 \)

Hence \( F \) is 1-1.
Since $\mathbb{RP}^2$ is compact and $F(\mathbb{RP}^2)$ is Hausdorff, $F$ is homeo. onto its image.

Hence $F: \mathbb{RP}^2 \to \mathbb{R}^4$ is a smooth embedding.

8-1 $\forall p=(x, y, s, t) \in F^{-1}(0,1)$

$$DF_p = \begin{pmatrix} 2x & 1 & 0 & 0 \\ 2x & 2y & 2s & 2t \end{pmatrix}$$

so $DF_p$ does not have rank 2 only when $y = s = t = 0$.

However $1 = x^2 + y^2 + s^2 + t^2 + y = x^2$ and $0 = x^2 + y = x^2$,

a contradiction, so $DF_p$ always has rank 2.

$\Rightarrow (0,1)$ is a regular value of $F$.

Note that $(x, y, s, t) \in F^{-1}(0,1)$ iff.

$$x^2 + y = 0 \text{ and } x^2 + y^2 + s^2 + t^2 + y = 1$$

which implies that $y^2 + s^2 + t^2 = 1$, and $-1 \leq y \leq 0$.

The map $\phi: F^{-1}(0,1) \to S^2$

$$(x, y, s, t) \mapsto (-\sin(x), y, s, t)$$

gives the required diffeo.
The Jacobian matrix of $F$ at $(x,y)$ is given by

$$
\begin{pmatrix}
3x^2 + y & x + 3y
\end{pmatrix}
$$

which has rank 0 only when $(x,y) \in F^{-1}(0) \cup F^{-1}(\frac{1}{27})$

Hence $\forall \ c \ real, \ c \neq 0, \frac{1}{27}, \ F^{-1}(c)$ is an embedded submanifold of $\mathbb{R}^2$.