Q1) (a) let $(\mathcal{U}, \overset{x}{\overset{\shortparallel}{\phi}})$ and $(V, \overset{Y}{\overset{\shortparallel}{\psi}})$ be 2 charts around p

then at p $(\phi(p)=0=\psi(p))$ or $(x(p)=0=Y(p))$,

$$g(Y)\frac{\partial}{\partial Y} = V = f(x)\frac{\partial}{\partial x} \quad \text{at } p$$

$$\Rightarrow \quad g(Y) = f(Y)\frac{\partial Y}{\partial x}$$

$\left(\text{apply } \frac{\partial}{\partial x}\right)$ $$\Rightarrow \quad \frac{\partial g}{\partial Y}\frac{\partial Y}{\partial x} = \frac{\partial f}{\partial x}\frac{\partial Y}{\partial x} + f \cdot \frac{\partial^2 Y}{\partial x^2}$$

Evaluate both sides at p, since $f(p)=0$,

we have $\frac{\partial g}{\partial Y} = \frac{\partial f}{\partial x}$ at p

//

(b) Suppose X has infinite zeros $\{x_n\} \subset S^1$, then $x_n \to p \in S^1$ (WLOG)

Choose a coord chart $(\mathcal{U}, \varphi)$ around p, then on $\mathcal{U}$,

$$X = f\frac{\partial}{\partial x} \quad \text{and} \quad f(p_n) = 0 \quad \text{where } p_n = \varphi(x_n)$$

by continuity $f(0) = 0$

By hypothesis, $\left.\frac{\partial f}{\partial x}\right|_{x=0} \neq 0$,

however $\left.\frac{\partial f}{\partial x}\right|_{x=0} = \lim_{p_n \to 0} \frac{f(p_n) - f(0)}{p_n - 0} = 0.$ contradiction.

2a) We know that $GL(3,\mathbb{R})$ is an open submfd of $M(3,\mathbb{R})$ and $O(3,\mathbb{R})$ is a submfd of $GL(3,\mathbb{R})$

Consider $F: O(3,\mathbb{R}) \to \mathbb{R}$ given by $F(A) = \det A$ for all $A \in O(3,\mathbb{R})$.

It is clear that $SO(3,\mathbb{R}) = F^{-1}(1)$ and 1 is a regular value of F.

Hence $SO(3,\mathbb{R})$ is a submfd of $O(3,\mathbb{R})$, hence of $M(3,\mathbb{R})$ //

2b) Consider $F: \mathbb{R}^6 \longrightarrow \mathbb{R}^3$ s.t. $F(x_1, x_2, x_3, p_1, p_2, p_3)$

$= (x_1^2 + x_2^2 + x_3^2,\ p_1^2 + p_2^2 + p_3^2,\ x_1 p_1 + x_2 p_2 + x_3 p_3)$

Let $L \subset T\mathbb{R}^3$ be the set of unit vectors tangent to the sphere, then $L = F^{-1}(1,1,0)$

$\forall \underset{\underset{(x,\ p)}{\parallel}}{(x_1, x_2, x_3, p_1, p_2, p_3)} \in F^{-1}(1,1,0)$

$$DF_{(x,p)} = \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2p_1 & 2p_2 & 2p_3 \\ p_1 & p_2 & p_3 & x_1 & x_2 & x_3 \end{pmatrix}$$

Since $\|x\| = \|p\| = 1$, the first two rows are linearly independent ;

$x = (x_1, x_2, x_3)$, $p = (p_1, p_2, p_3)$

since $x \cdot p = 0$, the third row cannot be in the span of the first 2 rows,

hence $DF_{(x,p)}$ has full rank. This means $(1,1,0)$ is a regular value

of F , and L is a smooth submfd //

2c) $\sum_{i=1}^{3} |z_i|^2 = 1$ and $\sum_{i=1}^{3} z_i^2 = 1$ iff. $\sum_{i=1}^{3} x_i = 1$ and $y_1 = y_2 = y_3 = 0$

where $z_j = x_j + i y_j$.

Define $F: \mathbb{R}^6 \longrightarrow \mathbb{R}^4$. by

$$F(z_1, z_2, z_3) = F(x_1, x_2, x_3, y_1, y_2, y_3) = \left(\sum_{i=1}^{3} x_i^2, y_1, y_2, y_3\right)$$

Note $F^{-1}(1,0,0,0) = \{\sum_{i=1}^{3} |z_i|^2 = 1\} \cap \{\sum_{i=1}^{3} z_i^2 = 1\}$

$\forall (z_1, z_2, z_3) \in F^{-1}(1,0,0,0)$

$$DF_{(z_1,z_2,z_3)} = \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which has full rank since $\sum_{i=1}^{3} x_i^2 = 1$

Hence $F^{-1}(1,0,0,0)$ is a submfd //

3-5,

(a) Say for $\sigma_N : S^3 \setminus \{N\} \to \mathbb{R}^3$, $\sigma_N(x_1, x_2, x_3, x_4) = \left(\frac{x_1}{1-x_4}, \frac{x_2}{1-x_4}, \frac{x_3}{1-x_4}\right)$

so if $e^{it} z^2 \neq i$, ie $y_2 \cos t + x_2 \sin t \neq 1$ where $z^2 = x_2 + i y_2$
$z^1 = x_1 + i y_1$

we have $\sigma_N(\gamma_z(t)) =$

$$\left(\frac{x_1 \cos t - y_1 \sin t}{1 - (y_2 \cos t + x_2 \sin t)}, \frac{y_1 \cos t + x_1 \sin t}{1 - (y_2 \cos t + x_2 \sin t)}, \frac{x_2 \cos t - y_2 \sin t}{1 - (y_2 \cos t + x_2 \sin t)} \right)$$

which is smooth as long as $\gamma_z(t) \in S^3 \setminus \{N\}$
$\Leftrightarrow e^{it} z^2 \neq i$.

Similarly we can check that $\sigma_S(\gamma_z(t))$ is also smooth. //

b, For $\gamma_z(t) = (e^{it} z^1, e^{it} z^2) \neq (0, i)$, we compute

that $$\frac{d}{dt}(\sigma_N(\gamma_z(t))) = \left(\frac{x_1 x_2 + y_1 y_2 - (x_1 \sin t + y_1 \cos t)}{(1 - y_2 \cos t - x_2 \sin t)^2}, \right.$$

$$\left. \frac{y_1 x_2 - x_1 y_2 + (x_1 \cos t - y_1 \sin t)}{(1 - y_2 \cos t - x_2 \sin t)^2}, \frac{x_2^2 + y_2^2 - (x_2 \sin t + y_2 \cos t)}{(1 - y_2 \cos t - x_2 \sin t)^2} \right)$$

If $\frac{d}{dt}(\sigma_N(\gamma_z(t))) \neq 0$, then we compute that.

$$(x_1 x_2 + y_1 y_2)^2 + (y_1 x_2 - x_1 y_2)^2 = (x_1 \sin t + y_1 \cos t)^2 + (x_1 \cos t - y_1 \sin t)^2.$$

$\Rightarrow \quad (x_1^2+y_1^2)(x_2^2+y_2^2-1)=0.$

If $x_1^2+y_1^2=0$, then $|z^1|^2+|z^2|^2=1 \Rightarrow x_2^2+y_2^2=1$ or $|e^{it}z^2|^2=1$

$\Rightarrow \quad x_2 \sin t + y_2 \cos t = 1$ or $\operatorname{Im}(e^{it}z^2)=1$

in view of the 3rd coordinate of $\frac{d}{dt}(\sigma_N \gamma_z(t))$

Now $|e^{it}z^2|=1$ and $\operatorname{Im}(e^{it}z^2)=1$

$\Rightarrow \quad e^{it}z^2 = i$

contradicts $\gamma_z(t) \in S^3 \setminus \{N\}$

Similarly we can prove $\frac{d}{dt}(\sigma_S(\gamma_z(t))) \neq 0$ //

7-2 Since $F(x,y,z)=F(-x,-y,-z)$, F descends to a map from $\mathbb{RP}^2$ to $\mathbb{R}^4$, that map is again denoted by F.

DF is clearly injective by looking at its Jacobian matrix

Suppose $F(x_1,y_1,z_1)=F(x_2,y_2,z_2)$ and $\sum x_i^2+y_i^2+z_i^2=1$.

Then $x_1^2-y_1^2=x_2^2-y_2^2$ and $x_1y_1=x_2y_2$

$\Rightarrow \quad \pm(x_1+iy_1)=x_2+iy_2$

By looking at the other two equation from $F(x_1,y_1,z_1)=F(x_2,y_2,z_2)$ we see that $\pm z_1 = z_2$

Hence F is 1-1

Since $\mathbb{RP}^2$ is cpt. and $F(\mathbb{RP}^2)$ is Hausdorff, F is homeo. onto its image.

Hence $F: \mathbb{RP}^2 \to \mathbb{R}^4$ is a smooth embedding //

8-1 $\forall p = (x, y, s, t) \in F^{-1}(0,1)$

$$DF_p = \begin{pmatrix} 2x & 1 & 0 & 0 \\ 2x & 2y+1 & 2s & 2t \end{pmatrix}$$

so DF_p does not have rank 2 only when $y = s = t = 0$,

however $1 = x^2 + y^2 + s^2 + t^2 + y = x^2$ and $0 = x^2 + y = x^2$,

a contradiction, so DF_p always has rank 2.

$\Rightarrow (0,1)$ is a regular value of F.

Note that $(x, y, s, t) \in F^{-1}(0,1)$ iff.

$x^2 + y = 0$ and $x^2 + y^2 + s^2 + t^2 + y = 1$ which implies

that $y^2 + s^2 + t^2 = 1$. and $-1 \le y \le 0$.

The map $\phi: F^{-1}(0,1) \longrightarrow S^2$

$(x, y, s, t) \longmapsto (-\mathrm{sign}(x)\, y, s, t)$

gives the required diffeo. //

8-2)

The Jacobian matrix of F at (x,y) is given by

$$\left[3x^2+y \quad x+3y^2\right)$$

which has rank 0 only when $(x,y) \in F^{-1}(0) \cup F^{-1}(\frac{1}{27})$

Hence $\forall c$ real, $c \neq 0, \frac{1}{27}$, $F^{-1}(c)$ is an embedded submfd of $\mathbb{R}^2$ //