2-1

Let σ_N, σ_S be the stereographic proj. from $S^n \setminus \{N\}$ and $S^n \setminus \{S\}$ resp.

a) The domain of definition for σ_N and σ_S are

$$S^1 \setminus \{N, e^{i\left(\frac{\pi + 4k\pi}{2n}\right)} \text{ where } 0 \le k \le n-1\},$$

$$S^1 \setminus \{S, e^{i\left(\frac{-\pi + 4k\pi}{2n}\right)} \text{ where } 0 \le k \le n-1\}$$

if we want $\alpha \circ p_n \circ \beta^{-1}$ to be smooth $(\alpha, \beta \in \{\sigma_N, \sigma_S\})$

For example $\sigma_S\, p_n\, \sigma_N^{-1}(u)$

$$= \sigma_S\left(\left(\frac{2u}{u^2+1} + i\,\frac{u^2-1}{u^2+1}\right)^n\right)$$

$$\triangleq \sigma_S\left(e^{i\theta}\right)^n$$

$$= \frac{\cos n\theta}{1 + \sin\theta}$$

smooth on the domain of definition.

2-1 (cont.)

b) For example,

$$\sigma_S \circ \sigma_N^{-1}(u_1, \cdots, u_n) = \sigma_S\left(\frac{-2u_1}{|u|^2+1}, \cdots, \frac{-2u_n}{|u|^2+1}, \frac{1-|u|^2}{1+|u|^2}\right)$$

$$= (-u_1, \cdots, -u_n)$$

smooth on the domain of def.

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c) For example,

$$\sigma_S F \sigma_N^{-1}(u_1, u_2, u_3) = \left(\frac{8u_1u_3 + 4u_2(|u|^2-1)}{8(u_1^2+u_2^2)}, \frac{8u_2u_3 - 4u_1(|u|^2-1)}{8(u_1^2+u_2^2)}\right)$$

Note that $F\sigma_N^{-1}(0, 0, v) = F\left(0, \frac{2v}{v^2+1} + i\frac{v^2-1}{v^2+1}\right)$

$= (0, 0, 1)$ which is not in the domain of def of σ_S

hence $\sigma_S F \sigma_N^{-1}$ is smooth on the domain of def.

2-4

$\tilde{P}$ well-defined:

For $\lambda \neq 0$, $x \neq 0$, $\underbrace{[P(\lambda x)]}_{\tilde{P}[\lambda x] \,=}$ $= [\lambda^d P(x)] = [P(x)] = \tilde{P}[x]$ ✓

$\tilde{P}$ smooth:

Let $U_i = \{ [x] \in \mathbb{RP}^n : x_i \neq 0 \}$

$V_j = \{ [y] \in \mathbb{RP}^k : y_j \neq 0 \}$

$\phi_i : U_i \to \mathbb{R}^n$, $\varphi_j : V_j \to \mathbb{R}^k$ charts are

$\mathbb{RP}^n$ and $\mathbb{RP}^k$ resp.

then $\varphi_j \circ \tilde{P} \circ \phi_i^{-1} (u_1, \cdots, u_n)$

$= \varphi_j \circ \tilde{P} \; [u_1 : \cdots : u_{i-1} : 1 : u_i : \cdots : u_n]$

$= \varphi_j [P_1(v) : \cdots : P_{k+1}(v)]$

where $v = (u_1, \cdots, u_{i-1}, 1, u_i, \cdots, u_n)$ and $P_j(v)$ is the j-component of $P(v)$

2-4 (cont.)

so $\varphi_j \circ \tilde{P} \circ \phi_i^{-1}(u_1, \cdots, u_n)$

$$= \left[\frac{P_1(v)}{P_j(v)} : \cdots : \widehat{\frac{P_j(v)}{P_j(v)}} : \cdots : \frac{P_{k+1}(v)}{P_j(v)} \right]$$

omitted

which is smooth since $P_j(v) \neq 0$ and P is smooth.

2-10

a) Let (U_i, φ_i) be a chart on $\mathbb{CP}^n$ s.t.

$$U_i = \{ [z_1 : \cdots : z_{n+1}] \in \mathbb{CP}^n \mid z_i \neq 0 \}$$

$\varphi_i : U_i \longrightarrow \mathbb{C}^n$ with $\varphi_i [z_1 : \cdots : z_{n+1}] =$

$$\left(\frac{z_1}{z_i}, \cdots \widehat{\frac{z_i}{z_i}}, \cdots, \frac{z_{n+1}}{z_i} \right)$$

now $\varphi_i \circ \pi (z_1, \cdots, z_{n+1}) = \left(\frac{z_1}{z_i}, \cdots, \widehat{\frac{z_i}{z_i}}, \cdots, \frac{z_{n+1}}{z_i} \right)$

which is Euclidean smooth by identifying

$x + iy \in \mathbb{C}$ with $(x,y) \in \mathbb{R}^2$.

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b) Consider $S^2 = \{ (z, r) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + r^2 = 1 \}$

Define $F : S^2 \longrightarrow \mathbb{CP}^1$ by $F(z, r) = \begin{cases} [z : 1-r] & \text{if } r \neq 1 \\ [1+r : \bar{z}] & \text{if } r \neq -1 \end{cases}$

Note if $r \neq \pm 1$, then $z\bar{z} = |z|^2 = 1 - r^2 = (1+r)(1-r)$

$$\Rightarrow \quad \frac{z}{1-r} = \frac{1+r}{\bar{z}} \qquad (z \neq 0 \text{ since } |r| \neq 1).$$

2-10 b) (cont.)

hence $[z : 1-r] = \left[\frac{z}{1-r} : 1\right] = \left[\frac{1+r}{\bar{z}} : 1\right] = [1+r : \bar{z}]$

F is the inverse of the map $G : \mathbb{CP}^1 \to S^2$ given by

$$G[z_1 : z_2] = \left(\frac{2 z_1 \bar{z}_2}{|z_1|^2 + |z_2|^2} , \frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2} \right).$$

hence F is bijective.

Now it remains to check F is a local diffeo.,

let $U_N = S^2 \setminus \{N\}$, $U_S = S^2 \setminus \{S\}$, σ_N, σ_S

stereographic proj. from N and S resp.,

let $\mathcal{U}_i = \{ [z_1 : z_2] \in \mathbb{CP}^1 \mid z_i \neq 0 \}$ $i = 1, 2$.

$\varphi_1 [z_1 : z_2] = \frac{z_2}{z_1}$ on $\mathcal{U}_1$, $\varphi_2 [z_1 : z_2] = \frac{z_1}{z_2}$ on $\mathcal{U}_2$.

check: $\varphi_2 \circ F \circ \sigma_N^{-1} (z) = z$

$\varphi_1 \circ F \circ \sigma_S^{-1} (z) = \bar{z}$

which are differ. from $\mathbb{C}$ to $\mathbb{C}$ (or from $\mathbb{R}^2$ to $\mathbb{R}^2$ via $x + iy \rightsquigarrow (x, y)$)

2-14

($\Leftarrow$) Take $p \in M$, choose V in the given open cover $\mathcal{U}$ st. $p \in V$, then by hypothesis V only intersect finitely many elts. in $\mathcal{U}$.

($\Rightarrow$) Take $V \in \mathcal{U}$. $\forall p \in M$, $\exists$ open nbd. A_p of p st. A_p only intersects finitely many elts. in $\mathcal{U}$.,

note $\bigcup_{p \in \bar{V}} A_p$ is an open cover for $\bar{V}$ which is cpt. by assumption, so there is a sub-cover, say $\bigcup_{i=1}^{n} A_{p_i}$.

For each A_{p_i}, let $U_{i,1}, \cdots, U_{i,n_i}$ be the elts. in $\mathcal{U}$ that has non-empty intersection with A_{p_i}.

Let $\Lambda = \{ U_{i,j} \mid 1 \le i \le n, 1 \le j \le n_i \} \subset \mathcal{U}$

If $\exists K \in \mathcal{U}$ st $V \cap K \neq \phi$, then $\exists A_{p_i}$ st. $K \cap A_{p_i} \neq \phi$, so $K \in \Lambda$

Hence V only intersects finitely many elts. in $\mathcal{U}$.

2-14 (cont.)

Suppose precompactness is dropped:

let $M = (0, \infty)$, $\mathcal{U} = \{ (n, \infty) : n \in \mathbb{N} \cup \{0\} \}$

then elts in $\mathcal{U}$ are not precompact, $\mathcal{U}$ is locally finite

but every elt in $\mathcal{U}$ intersect all other elts. in $\mathcal{U}$.

Suppose openness is dropped:

M any smooth mfd., $\mathcal{U} = \{ \{p\} \mid p \in M \}$

then $\mathcal{U}$ isn't locally finite, but every $\{p\} \in \mathcal{U}$ only intersect with itself.