Let $\sigma_N$, $\sigma_S$ be the stereographic proj. from $S^n \setminus \{N\}$ and $S^n \setminus \{S\}$ respectively.

9) The domain of definition for $\sigma_N$ and $\sigma_S$ are $S^1 \setminus \{N\}$, $e^{i \left( \frac{\pi + 4k\pi}{2n} \right)}$ where $0 \leq k \leq n-1$, $S^1 \setminus \{S\}$, $e^{i \left( \frac{-\pi + 4k\pi}{2n} \right)}$ where $0 \leq k \leq n-1$.

If we want $\log \circ \beta^r$ to be smooth ($\beta \in \{\sigma_N, \sigma_S\}$).

For example:

$\sigma_S \circ \sigma_N^{-1} (u)$

$= \sigma_S \left( \left( \frac{2u}{u^2 + 1} + i \frac{u^2 - 1}{u^2 + 1} \right)^n \right)$

$\triangleq \sigma_S \left( e^{i \theta} \right)^n$

$= \frac{\cos n\theta}{1 + \sin n\theta}$

Smooth on the domain of definition.
b) For example,

$$\sigma_S = \sigma_N^{-1}(u_1, \ldots, u_n) = \sigma_S \left( \frac{-2u_1}{1u_1^2 + 1}, \ldots, \frac{-2u_n}{1u_n^2 + 1}, \frac{1 - 1u^2}{1 + 1u^2} \right) = (-u_1, \ldots, -u_n)$$

smooth on the domain of def.

---

c) For example,

$$\sigma_S \sigma_N^{-1}(u_1, u_2, u_3) = \left( \frac{8u_1 u_3 + 4u_2 (1u_1^2 - 1)}{8 (u_1^2 + u_2^2)}, \frac{8u_2 u_3 - 4u_1 (1u_1^2 - 1)}{8 (u_1^2 + u_2^2)} \right)$$

Note that $\sigma_N^{-1}(0, 0, 0) = F(0, \frac{2u}{V^2_t + 1}, i \frac{V^2}{V^2_t})$

$$= (0, 0, 0), \text{ which is not in the domain of def of } \sigma_S$$

hence $\sigma_S \sigma_N^{-1}$ is smooth on the domain of def.
\( \bar{P} \) well-defined: \( \bar{P}(x) = \frac{1}{n} \)

For \( \lambda \neq 0, \ x \neq 0 \):

\[ [P(\lambda x)] = [A^d P(x)] = [P(x)] = \bar{P}(x) \]

\( \bar{P} \) smooth:

Let \( U_i = \{ x \in \mathbb{R} \mathbb{P}^n : x_i \neq 0 \} \)

\[ V_j = \{ y \in \mathbb{R} \mathbb{P}^k : y_j \neq 0 \} \]

\( \phi_i : U_i \to \mathbb{R}^n, \ \psi_j : V_j \to \mathbb{R}^k \) charts are \( \mathbb{R} \mathbb{P}^n \) and \( \mathbb{R} \mathbb{P}^k \) resp.

Then

\[ \psi_j \circ \bar{P} \circ \phi_i^{-1} (u_1, \ldots, u_n) \]

\[ = \psi_j \circ \bar{P} \left[ u_1, \ldots, u_i^{-1}, 1, u_{i+1}, \ldots, u_n \right] \]

\[ = \psi_j \left[ P_1(v) : \ldots : P_{k+1}(v) \right] \]

where \( v = (u_1, \ldots, u_i^{-1}, 1, u_{i+1}, \ldots, u_n) \) and \( P_j(v) \) is the \( j \)-th component of \( P(v) \).
so \( \psi_j \circ \widetilde{\text{P}} \circ \phi_i^{-1} (u, \ldots, u_n) \)

\[
= \begin{bmatrix}
\frac{P_1(v)}{P_j(v)} & \ldots & \frac{P_j(v)}{P_j(v)} & \ldots & \frac{P_k(v)}{P_j(v)} \\
\frac{P_j(v)}{P_j(v)} & \ldots & \frac{P_j(v)}{P_j(v)} & \ldots & \frac{P_j(v)}{P_j(v)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{P_j(v)}{P_j(v)} & \ldots & \frac{P_j(v)}{P_j(v)} & \ldots & \frac{P_j(v)}{P_j(v)} \\
\end{bmatrix}
\]

\[
\uparrow
\]

omitted

which is smooth since \( P_j(v) \neq 0 \) and \( P \) is smooth.
a) Let \( (U_i, \varphi_i) \) be a chart on \( \mathbb{CP}^n \) st.

\[
U_i = \{ [z_0 : \ldots : z_{n+1}] \in \mathbb{CP}^n \mid z_i \neq 0 \}
\]

\[
\varphi_i : U_i \rightarrow \mathbb{C}^n \quad \text{with} \quad \varphi_i [z_0 : \ldots : z_{n+1}] = \left( \frac{\partial}{\partial \bar{z}_i}, \ldots, \frac{\partial}{\partial \bar{z}_i}, \ldots, \frac{\partial}{\partial \bar{z}_i} \right)
\]

now \( \varphi_i \circ \pi (z_0 : \ldots : z_{n+1}) = \left( \frac{\partial}{\partial \bar{z}_i}, \ldots, \frac{\partial}{\partial \bar{z}_i}, \ldots, \frac{\partial}{\partial \bar{z}_i} \right) \)

which is Euclidean smooth by identifying

\( x + iy \in \mathbb{C} \) with \((x, y) \in \mathbb{R}^2\).

b) Consider \( S^2 = \{ (z, r) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + r^2 = 1 \} \)

Define \( F : S^2 \rightarrow \mathbb{CP}^1 \) by

\[
F(z, r) = \begin{cases} 
[\bar{z} : 1-r] & \text{if } r \neq 1 \\
[1+r : \bar{z}] & \text{if } r = 1 
\end{cases}
\]

Note if \( r \neq 1 \), then \( \bar{z} \bar{z} = |z|^2 = 1-r^2 = (1+r)(1-r) \)

\[
\Rightarrow \frac{\bar{z}}{1-r} = \frac{1+r}{\bar{z}} \quad (\bar{z} \neq 0 \text{ and } |1+r| \neq 1).
\]
Here $[g : 1-r] = \left[ \frac{-r}{1-r} : 1 \right] = \left[ \frac{1+r}{g} : 1 \right] = \left[ 1+r : g \right]$

$F$ is the inverse of the map $G : \mathbb{CP}^1 \to S^3$ given by

$G \left[ g_1 : g_2 \right] = \left( \frac{2 g_1 \overline{g_2}}{g_1^2 + g_2^2}, \frac{4 g_1^2 - 1}{g_1^2 + g_2^2}, \frac{4 g_2^2 - 1}{g_1^2 + g_2^2} \right)$

hence $F$ is bijective.

Now it remains to check $F$ is a local diffeo.

Let $U_N = S^3 \setminus \{N_3\}$, $U_S = S^3 \setminus \{S\}$, $\sigma_N$, $\sigma_S$ stereographic proj. from $N$ and $S$ resp.

Let $U_i = \left\{ \left[ g_1 : g_2 \right] \in \mathbb{CP}^1 \mid \overline{g}_i \neq 0 \right\}$, $i = 1, 2$.

$\varphi_1 \left[ g_1 : g_2 \right] = \frac{g_1}{g_2}$ on $U_1$, $\varphi_2 \left[ g_1 : g_2 \right] = \frac{g_2}{g_1}$ on $U_2$.

Check:

$\varphi_2 \circ F \circ \varphi_N^{-1} (g) = \overline{g}$

$\varphi_1 \circ F \circ \varphi_S^{-1} (g) = \overline{g}$

which are diffeo. from $C$ to $C$ (or from $\mathbb{R}^2$ to $\mathbb{R}^2$) via $x+iy \to (x,y)$. 
(\leq) \text{ Take } p \in M, \text{ choose } V \text{ in the given open cover } U \text{ st. } p \in V, \text{ then by hypothesis } V \text{ only intersects finitely many elts. in } U.

(\Rightarrow) \text{ Take } V \in U, \forall p \in M, \exists \text{ open nbd. } A_p \text{ of } p \text{ st. } A_p \text{ only intersects finitely many elts. in } U, \text{ note } U A_p \text{ is an open cover for } V \text{ which is rel. by assumption, so there is a sub-cover, say } \bigcup_{p \in V} A_p, \text{ for each } A_p, \text{ let } U_{i_1}, \ldots, U_{i_{n_k}} \text{ be the elts. in } U \text{ that has non-empty intersection with } A_p. \text{ Let } \Lambda = \{ U_{i_1}, \ldots, U_{i_{n_k}} \} \subset U. \text{ If } \exists K \in U \text{ st. } V \cap K \neq \emptyset, \text{ then } \exists A_p \text{ st. } K \cap A_p \neq \emptyset, \text{ so } K \in \Lambda. \text{ Hence } V \text{ only intersects finitely many elts. in } U.
Suppose precompactness is dropped:

\[ M = (0, \infty), \quad U = \{ (n, \infty) : n \in \mathbb{N} U \{0\} \} \]

The els in U are not precompact. U is locally finite, but every elt in U interfered all other els. in U.

Suppose openness is dropped:

M any smooth mfd., \[ U = \{ \{p^3 \mid p \in M \} \]  

Then U not locally finite, but every \( \{p^3 \} \in U \) only interfered with itself.