HW1

1) $$dF_{(x,y)} = \begin{pmatrix} \cos(x+y) & 2x \\ \cos(x+y) & -2y \end{pmatrix}$$

$$\det\left(dF_{(x,y)}\right) = 0$$

$$\Leftrightarrow \quad (x+y)\cos(x+y) = 0$$

$$\Leftrightarrow \quad x+y=0 \quad \text{or} \quad x+y = k\pi + \tfrac{\pi}{2},\ k \in \mathbb{Z}$$

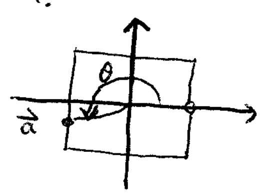
i.e. F is a local diffeo. on $\mathbb{R}^2 \setminus \{(x,y)\in\mathbb{R}^2 \mid x+y=0 \text{ or } x+y=k\pi+\frac{\pi}{2}\}$

(There are many possible answers to this problem)

2) $M = \{(x_1,x_2)\in\mathbb{R}^2 \mid \max\{|x_1|,|x_2|\}=1\}$

Let $U_1 = M\setminus\{(1,0)\}$ $\qquad U_2 = M\setminus\{(-1,0)\}$.

Define $\varphi_1 : U_1 \to (0, 2\pi)$



$\vec{a}=(x,y) \longmapsto \theta =$ angle of $\vec{a}$ with +ve x-axis measured counter-clockwisely

$\varphi_2 : U_2 \longrightarrow (-\pi, \pi)$

$$\begin{matrix}(x,y)\\ \shortparallel \\ x+iy\end{matrix} \longmapsto \operatorname{Arg}(x+iy)$$ where $\operatorname{Arg}(z)$ is the Principle Argument of $z\in\mathbb{C}$

It is then straight forward to check φ_1 and φ_2 overlap smoothly.

Problem 1-1

$[(x,y)]$ denotes an equivalento class

M is locally Euclideant 2nd countable:

Let $\mathcal{U}_1 = \{ [(x,1)] \in X \mid x \neq 0 \} \cup \{ [(0,1)] \}$

$\mathcal{U}_2 = \{ [(x,1)] \in X \mid x \neq 0 \} \cup \{ [(0,-1)] \}$

$\mathcal{U}_1$ and $\mathcal{U}_2$ are naturally homeo to $\mathbb{R}$, so they are 2nd countable and locally Euclidean, hence $M = \mathcal{U}_1 \cup \mathcal{U}_2$ is also 2nd countable and locally Euclidean.

M is not Hausdorff:

If A + B are open sets in M containing $[(0,1)]$ + $[(0,-1)]$ resp., then for $\varepsilon > 0$ small enough, $[(\varepsilon, 1)] \in A \cap B$

Problem 1-4

WLOG, let $\min(m,n) = n$, let $A_k = \{ A \in M_{m \times n}(\mathbb{R}) \mid \operatorname{rank} A \geq k \}$

For all A in $M_{m \times n}(\mathbb{R})$, let $\sigma(A) = \{ k$ by k sub-matrices of $A \}$

note that $|\sigma(A)| < \infty$, hence the fct. (function)

$$f : M_{m \times n}(\mathbb{R}) \longrightarrow \mathbb{R}$$

defined by $f(A) = \sum_{\alpha \in \sigma(A)} |\det(\alpha)|$ is a well-defined cts. fct.

Since $A_k = f^{-1}((0,\infty))$, it is an open subset of $M_{m\times n}(\mathbb{R})$, hence an open submfd.

Not true if "at least k" is replaced by "equal to k":

For example $m=n=2$, $k=1$

let $X = \{A \in M_2(\mathbb{R}) \mid \operatorname{rank} A = 1\}$,

but $M_2(\mathbb{R}) \setminus X = \{A \in M_2(\mathbb{R}) \mid \operatorname{rank} A = 0, 2\}$ is not closed,

it is because $\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} \in M_2(\mathbb{R}) \setminus X$ for all ε,

but $\lim_{\varepsilon\to 0} \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in X$.

Problem 1-5

a) for $x \in S^n \setminus \{N\}$, consider the straight line $\gamma(t) = t(x-N)+N \in \mathbb{R}^{n+1}$

Note that $\gamma\left(\frac{1}{1-x_{n+1}}\right) = \left(\frac{x_1}{1-x_{n+1}}, \cdots, \frac{x_n}{1-x_{n+1}}, 0\right)$ which lies on $\{x_{n+1} = 0\}$ and $\gamma\left(\frac{1}{1-x_{n+1}}\right) \cong \sigma(x)$

For $x \in S^n \setminus \{S\}$, consider the straight line $\beta(t) = t(x-S)+S \in \mathbb{R}^{n+1}$

$$\beta\left(\frac{1}{1+x_{n+1}}\right) = \left(\frac{x_1}{1+x_{n+1}}, \cdots, \frac{x_n}{1+x_{n+1}}, 0\right) \simeq \tilde{\sigma}(x) = -\sigma(-x)$$

b) Suppose $\sigma(x_1, \cdots, x_{n+1}) = (u_1, \cdots, u_n)$

$$\Rightarrow \frac{x_i}{1-x_{n+1}} = u_i \qquad \forall\, 1 \le i \le n.$$

Use $\sum_{i=1}^{n+1} x_i^2 = 1$, we get $\frac{1-x_{n+1}^2}{(1-x_{n+1})^2} = \sum_{i=1}^{n} u_i^2 = |u|^2$.

$$\Rightarrow x_{n+1} = \frac{|u|^2-1}{|u|^2+1}$$

$$\Rightarrow x_i = \frac{2u_i}{|u|^2+1}, \quad 1 \le i \le n$$

$$\Rightarrow \sigma^{-1}(u_1, \cdots, u_n) = \left(\frac{2u_1}{|u|^2+1}, \cdots, \frac{2u_n}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1}\right)$$

Since the formula for σ^{-1} is obtained, σ is then bijective.

c) Two charts $(S^n \setminus \{N\}, \sigma)$, $(S^n \setminus \{S\}, \tilde{\sigma})$ intersect on $S^n \setminus \{N, S\}$, so the function $\tilde{\sigma} \circ \sigma^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ given by $\tilde{\sigma} \circ \sigma^{-1}(u) = \frac{u}{|u|^2}$ is smooth.

d) For simplicity, let's first check $(S^n \setminus \{N\}, \sigma)$ and $(U_{n+1}^+, \varphi_{n+1}^+)$ overlap smoothly.

$$\sigma \circ (\varphi_{n+1}^+)^{-1}(u_1, \cdots, u_n) = \sigma(u_1, \cdots, u_n, \sqrt{1-|u|^2}) \quad \text{where } u = (u_1, \cdots, u_n)$$

$$= \frac{u}{1-\sqrt{1-|u|^2}}$$ which is smooth if $u \neq 0$, but $u = 0$ is not in $\varphi_{n+1}^+(U_{n+1}^+ \cap (S^n \setminus \{N\}))$

$$\varphi_{n+1}^+ \circ \sigma^{-1}(u) = \varphi_{n+1}^+\left(\frac{2u_1}{|u|^2+1}, \cdots, \frac{2u_n}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1}\right)$$

$$= \left(\frac{2u_1}{|u|^2+1}, \cdots, \frac{2u_n}{|u|^2+1}\right)$$ which is also smooth.

Also check that $(S^n \setminus \{N\}, \sigma)$ and (U_i^+, φ_i^+) overlap smoothly for $1 \le i \le n$.

$$\sigma \circ (\varphi_i^+)^{-1}(u_1, \cdots, u_n) = \sigma(u_1, \cdots, u_i, \sqrt{1-|u|^2}, u_{i+1}, \cdots, u_n)$$

$$= \left(\frac{u_1}{1-u_n}, \cdots, \frac{u_i}{1-u_n}, \frac{\sqrt{1-|u|^2}}{1-u_n}, \frac{u_{i+1}}{1-u_n}, \cdots, \frac{u_{n-1}}{1-u_n}\right)$$

smooth on $\varphi^+(U_i^+ \cap (S^n \setminus \{N\}))$

$$\varphi_i^+ \circ \sigma^{-1}(u) = \varphi_i^+\left(\frac{2u_1}{1+|u|^2}, \cdots, \frac{2u_i}{1+|u|^2}, \cdots, \frac{2u_n}{1+|u|^2}, \frac{|u|^2-1}{|u|^2+1}\right)$$

$$= \left(\frac{2u_1}{1+|u|^2}, \cdots, \frac{2u_{i-1}}{1+|u|^2}, \frac{2u_{i+1}}{1+|u|^2}, \cdots, \frac{2u_n}{1+|u|^2}, \frac{|u|^2-1}{|u|^2+1}\right)$$ is smooth.

The rest of the chart can be checked similarly.

Problem 1-7

Manifold structure on $\mathbb{CP}^n$:

Let $U_i = \{ [x_0 : x_1 : \cdots : x_n] \in \mathbb{CP}^n \mid x_i \neq 0 \}$ where $[x_0 : \cdots : x_n]$ is the homogeneous coordinate on $\mathbb{CP}^n$.

Define $\varphi_i : U_i \longrightarrow \mathbb{R}^{2n}$

$$[x_0 : x_1 : \cdots : x_n] \longmapsto \left(\frac{x_0}{x_i}, \cdots, \widehat{\frac{x_i}{x_i}}, \cdots, \frac{x_n}{x_i} \right)$$ where $\wedge$ means that term is omitted.

$\{U_i\}_{i=0}^{n}$ is an open cover for $\mathbb{CP}^n$.

If $\underset{(\text{say } l<j)}{l \neq j}$, check $\varphi_l \circ \varphi_j^{-1} : \varphi_j(U_l \cap U_j) \longrightarrow \varphi_l(U_l \cap U_j)$ is smooth,

$$\varphi_l \circ \varphi_j^{-1}(x_1, y_1, \cdots, x_n, y_n) = \varphi_l([x_1 + iy_1 : \cdots : x_{j-1} + iy_{j-1} : 1 : x_{j+1} + iy_{j+1} : \cdots : x_n + iy_n])$$

(let $z_k = x_k + iy_k$)

$$= \left(\frac{z_1}{z_l}, \cdots, \frac{z_{l-1}}{z_l}, 1, \frac{z_{l+1}}{z_l}, \cdots, \frac{z_{j-1}}{z_l}, \frac{1}{z_l}, \frac{z_{j+1}}{z_l}, \cdots, \frac{z_n}{z_l} \right)$$ is

smooth since $z_l \neq 0$ on $U_l \cap U_j$

By symmetry on l and j, $\varphi_j \circ \varphi_l^{-1}$ is the smooth inverse of $\varphi_l \circ \varphi_j^{-1}$.

$\mathbb{CP}^n$ is 2nd countable and Hausdorff because its topology is given by the quotient topology from $\mathbb{C}^{n+1}$.

It is clear that $\dim \mathbb{CP}^n = 2n$.

$\mathbb{CP}^n$ is cpt.:

Because $\mathbb{CP}^n = S^n_{\mathbb{C}}/\sim$ with quotient topology

where $S^n_{\mathbb{C}} = \{ (z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum_{i=1}^{n+1} |z_i|^2 = 1 \}$

and $(z_1, \cdots, z_{n+1}) \sim (w_1, \cdots, w_{n+1}) \iff (z_1, \cdots, z_{n+1}) = \pm (w_1, \cdots, w_{n+1})$

Since $S^n_{\mathbb{C}}$ is cpt, so as $\mathbb{CP}^n$.