Math 518 Differentiable Manifolds I

Some Practice Problems

The take-home midterm will be distributed on Thursday Oct.28 and is due at the start of class (11:00AM sharp) on Tuesday Nov.2. Below are some practice problems.

1. If $F\colon M \to N$ is a submersion and $U \subset M$ is open, show that $F(U)$ is open in N.

2. Let $M(n)$ be the space of all $n \times n$-matrices with real entries. Show that any nonzero number is a regular value of the determinant map $\det\colon M(n) \to \mathbb{R}$. (Hint: Compute $(D\det)_A(A)$ for $\det A \neq 0$.)

3. Prove that the set of points
$$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1,\ x_1x_2x_3 = 1\}$$
is a smooth submanifold of $\mathbb{R}^4$.

4. (a) Verify that the map
$$t \mapsto \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2}\right)$$
is an embedding of $\mathbb{R}$ into $\mathbb{R}^2$.
   (b) Determine if the map $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by
$$F(x, y) = (x\cos y, x\sin y, x)$$
is an immersion.
   (c) Check that the quotient map $q\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ is a submersion.

5. If M is compact and N is connected show that every submersion $F\colon M \to N$ is surjective.

6. Let M be a compact manifold and assume that $\dim M = \dim N$. Let $y \in N$ be a regular value of a smooth mapping $F\colon M \to N$. Show that $F^{-1}(y)$ is a finite set $\{x_1, \ldots, x_k\}$. Prove that there exists a neighborhood V of y in N such that $F^{-1}(V)$ is a disjoint union $U_1 \cup \ldots \cup U_k$ where each U_j is an open neighborhood of x_j and $F|_{U_j}$ is a diffeomorphism onto V.

7. For which values of a does the hyperboloid $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversally? What do the intersections look like for different values of a?

8. Let V be a vector space and let Δ be the diagonal in $V \times V$. For a linear map $A\colon V \to V$ consider the graph

$$W = \{(v, Av) \mid v \in V\}.$$

Show that $W \pitchfork \Delta$ iff 1 is not an eigenvalue of A.

9. Let V and W be subspaces of $\mathbb{R}^n$, and let

$$q\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}P^{n-1}$$

be the usual quotient map.

   (a) Prove that for any subspace U of $\mathbb{R}^n$, $q(U \setminus \{0\})$ is a submanifold of $\mathbb{R}P^{n-1}$ of dimension equal to $\mathrm{dim(U)} - 1$

   (b) Show that $V \pitchfork W$ iff $\mathrm{dim(V) + dim(W) - dim(V \cap W) = n}$.

   (c) Show that if $V \pitchfork W$ then $q(V \setminus \{0\}) \pitchfork q(W \setminus \{0\})$.

10. Suppose that M is a submanifold of $\mathbb{R}^n$. Show that "almost every" vector subspace V of $\mathbb{R}^n$ with fixed dimension l, intersects M transversally. Hint: The subset $S \subset (\mathbb{R}^n)^l$ consisting of all linear independent l-tuples of vectors in $\mathbb{R}^n$ is open, and the map $\mathbb{R}^l \times S \to \mathbb{R}^n$ defined by

$$[(t_1, \ldots, t_l), v_1, \ldots, v_l] \mapsto t_1 v_1 + \cdots + t_l v_l$$

is a submersion.