Math 518 Differentiable Manifolds I

Some Practice Problems

The take-home midterm will be distributed on Thursday Oct. 28 and is due at the start of class (11:00AM sharp) on Tuesday Nov. 2. Below are some practice problems.

1. If $F: M \rightarrow N$ is a submersion and $U \subset M$ is open, show that $F(U)$ is open in $N$.

2. Let $M(n)$ be the space of all $n \times n$-matrices with real entries. Show that any nonzero number is a regular value of the determinant map $\det: M(n) \rightarrow \mathbb{R}$. (Hint: Compute $(D \det)_A(A)$ for $\det A \neq 0$.)

3. Prove that the set of points
   
   $$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1, x_1 x_2 x_3 = 1\}$$

   is a smooth submanifold of $\mathbb{R}^4$.

4. (a) Verify that the map
   
   $$t \mapsto \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2}\right)$$

   is an embedding of $\mathbb{R}$ into $\mathbb{R}^2$.

   (b) Determine if the map $F: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by
   
   $$F(x, y) = (x \cos y, x \sin y, x)$$

   is an immersion.

   (c) Check that the quotient map $q: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}P^n$ is a submersion.

5. If $M$ is compact and $N$ is connected show that every submersion $F: M \rightarrow N$ is surjective.

6. Let $M$ be a compact manifold and assume that $\dim M = \dim N$. Let $y \in N$ be a regular value of a smooth mapping $F: M \rightarrow N$. Show that $F^{-1}(y)$ is a finite set $\{x_1, \ldots, x_k\}$. Prove that there exists a neighborhood $V$ of $y$ in $N$ such that $F^{-1}(V)$ is a disjoint union $U_1 \cup \ldots \cup U_k$ where each $U_j$ is an open neighborhood of $x_j$ and $F|_{U_j}$ is a diffeomorphism onto $V$. 
7. For which values of $a$ does the hyperboloid $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversally? What do the intersections look like for different values of $a$?

8. Let $V$ be a vector space and let $\Delta$ be the diagonal in $V \times V$. For a linear map $A : V \to V$ consider the graph

$$W = \{(v, Av) \mid v \in V\}.$$ 

Show that $W \cap \Delta$ iff 1 is not an eigenvalue of $A$.

9. Let $V$ and $W$ be subspaces of $\mathbb{R}^n$, and let 

$$q : \mathbb{R}^n \setminus \{0\} \to \mathbb{RP}^{n-1}$$

be the usual quotient map.

(a) Prove that for any subspace $U$ of $\mathbb{R}^n$, $q(U \setminus \{0\})$ is a submanifold of $\mathbb{RP}^{n-1}$ of dimension equal to $\dim(U) - 1$.

(b) Show that $V \cap W$ iff $\dim(V) + \dim(W) - \dim(V \cap W) = n$.

(c) Show that if $V \cap W$ then $q(V \setminus \{0\}) \cap q(W \setminus \{0\})$.

10. Suppose that $M$ is a submanifold of $\mathbb{R}^n$. Show that ”almost every” vector subspace $V$ of $\mathbb{R}^n$ with fixed dimension $l$, intersects $M$ transversally.

Hint: The subset $S \subset (\mathbb{R}^n)^l$ consisting of all linear independent $l$-tuples of vectors in $\mathbb{R}^n$ is open, and the map $\mathbb{R}^l \times S \to \mathbb{R}^n$ defined by

$$[(t_1, \ldots, t_l), v_1, \ldots, v_l] \mapsto t_1 v_1 + \cdots + t_l v_l$$

is a submersion.