1) $O(q) = \{ A \in GL(n,\mathbb{R}) \mid q(Ax) = q(x) \quad \forall x \in \mathbb{R}^n \}$

$= \{ A \in GL(n,\mathbb{R}) \mid x^T A^T \Sigma A x = x^T \Sigma x \quad \forall x \in \mathbb{R}^n \}$

$= \{ A \in GL(n,\mathbb{R}) \mid A^T \Sigma A = \Sigma$

$= F^{-1}(\Sigma)$

where $F : GL(n,\mathbb{R}) \to S(n)$ is the smooth map

$$A \mapsto A^T \Sigma A$$

- It suffices to show that Σ is a regular value of F.

That is $\forall A \in F^{-1}(\Sigma)$ the map $F_* : T_A GL(n,\mathbb{R}) \to T_\Sigma S(n)$ is surjective.

- First we compute $F_*(v)$ for $v \in T_A GL(n,\mathbb{R})$

$v = [A + tB]$ for some $B \in M(n,\mathbb{R})$

- So $F_*(v) = F_*([A+tB])$

$= [F(A+tB)]$

$= [(A^T + tB^T)\Sigma(A+tB)]$

$$= [\, A^T \Sigma A + t(A^T \Sigma B + B^T \Sigma A) + t^2 B^T \Sigma B \,]$$

$$= [\, \Sigma + t(A^T \Sigma B + B^T \Sigma A) \,]$$

So $F_*([A + tB]) = [\Sigma + t(A^T \Sigma B + B^T \Sigma A)]$

- For any $w \in T_\Sigma S(n)$ we have

  $w = [\Sigma + tC]$ for some $C \in S(n)$

- Need to find $B \in M(n, \mathbb{R})$ such that

  $A^T \Sigma B + B^T \Sigma A = C.$

- Choosing $B = \frac{1}{2} A \Sigma^{-1} C$ we get

$$A^T \Sigma B + B^T \Sigma A = \frac{1}{2} A^T \Sigma A \Sigma^{-1} C + \frac{1}{2} C^T \Sigma^{-1} A^T \Sigma A$$

$$= \frac{1}{2} \Sigma \Sigma^{-1} C + \frac{1}{2} C^T \Sigma^{-1} \Sigma$$

$$= \frac{1}{2} C + \frac{1}{2} C^T$$

$$= C \quad (\text{since } C = C^T)$$

$\mathbb{Z}$

2) Choose a smooth chart (U, φ) around p and consider

$$\hat{F} = \varphi \circ F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$$

- To use Hint 1 we must show that

i) p is an isolated fixed point of F

$\Leftrightarrow$ $\varphi(p)$ is an isolated fixed point of $\tilde{F}$

ii) 1 is an eigenvalue of F_*

$\Leftrightarrow$ 1 is an eigenvalue of $\tilde{F}_* = D\hat{F}_{\varphi(p)}$.

Fact (i) is easy to show.

Fact (ii) follows from the obvious equivalence

$$F_*(v) = v \iff \varphi_* \circ F_* (\varphi^{-1})_* (\varphi_*(v)) = \varphi_*(v)$$

$$\iff \hat{F}_* (\varphi_*(v)) = \varphi_*(v).$$

- Hint 2 implies that

V is an open nbhd of $\ell(p)$ such that the only fixed point of $\tilde{F}$ in V is $\ell(p)$.

$$\Updownarrow$$

V is an open nbhd of $\ell(p)$ such that the only point in V which get mapped by $\tilde{F} - I_m$ to 0, is $\ell(p)$.

- Now $D(\tilde{F} - I_m)_{\ell(p)} = D\tilde{F}_{\ell(p)} - I_m$

$$\Rightarrow \det\left(D(\tilde{F} - I_m)_{\ell(p)}\right) = \det\left(D\tilde{F}_{\ell(p)} - I_m\right).$$

$\Rightarrow D(\tilde{F} - I_m)_{\ell(p)}$ is nonsingular $\Leftrightarrow$ 1 is not an eigenvalue of $D\tilde{F}_{\ell(p)}$

So overall, we have.

$F_* : T_pM \to T_pM$ does not have 1 as an eigenvalue.

$$\Updownarrow$$

$D\hat{F}_{\varphi(p)}$ does not have one as an eigenvalue.

$$\Updownarrow$$

$D(\hat{F}-I_m)_{\varphi(p)}$ is nonsingular.

$$\Downarrow \quad \text{(By Inverse Function Thm)}$$

$\hat{F}-I_m$ is a local diffeo near $\varphi(p)$.

$$\Downarrow$$

there is an open nbhd V of $\varphi(p)$ such that the only point in V which gets mapped by $\hat{F}-I_m$ to 0 is $\varphi(p)$

$$\Updownarrow$$

$\varphi(p)$ is the only fixed point of $\tilde{F} - I_n$ in V

$$\Downarrow$$

p is an isolated fixed point of F.

$\mathcal{Z}$

3a) $F : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}$

$$(z_1, z_2) \mapsto z_1^2 + z_2^3$$

Identify $\mathbb{C}^2 \setminus \{0\}$ with $\mathbb{R}^4 \setminus \{0\}$ by setting

$z_1 = x_1 + i y_1$ and $z_2 = x_2 + i y_2$.

Similarly, identify $\mathbb{C}$ with $\mathbb{R}^2$.

In these terms we have

$$F(x_1, y_1, x_2, y_2) = \left(x_1^2 - y_1^2 + x_2^3 - 3x_2 y_2^2,\ 2x_1 y_1 + 3x_2^2 y_2 - y_2^3\right)$$

Then

$$F_* = DF_p = \begin{bmatrix} 2x_1 & -2y_1 & 3(x_2^2 - y_2^2) & -6x_2 y_2 \\ 2y_1 & 2x_1 & 6x_2 y_2 & 3(x_2^2 - y_2^2) \end{bmatrix}$$

for $p = (x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \setminus \{0\}$.

Now p is a critical point of F iff the rows of DF_p are linearly dependent.

That is, $p = (x_1, y_1, x_2, y_2)$ is a critical point of F iff for some $c \in \mathbb{R}$ we have.

$$\begin{cases} 2x_1 = c2y_1 & \textcircled{1} \\ -2y_1 = c2x_1 & \textcircled{2} \\ 3(x_2^2 - y_2^2) = c6x_2y_2 & \textcircled{3} \\ -6x_2y_2 = c3(x_2^2 - y_2^2) & \textcircled{4} \end{cases}$$

$\textcircled{1} + \textcircled{2} \Rightarrow x_1 = y_1 = 0$

$\textcircled{3} + \textcircled{4} \Rightarrow x_2 = y_2 = 0$

So there are no critical points of F in $\mathbb{R}^4 \setminus \{0\}$.

Every point in $\mathbb{R}^4 \setminus \{0\}$ is a regular point and evry point in $\mathbb{R}^2$ is a regular value of F.

Z

3b) To show that $S^3 \pitchfork F^{-1}(0)$ we must show that for each $p \in S^3 \cap F^{-1}(0)$ we have

$$T_p S^3 + T_p F^{-1}(0) = T_p(\mathbb{R}^4 \setminus \{0\}) \qquad (*)$$

- Recall that if q is a regular value of $H: M \to N$ and $p \in H^{-1}(q)$ then

$$T_p H^{-1}(q) = \text{Kernel of } H_*: T_pM \to T_qN$$

Hence, for $p = (x_1, y_1, x_2, y_2)$ in $F^{-1}(0)$ the tangent space $T_p F^{-1}(0)$ is the kernel of the map.

$$F_* = DF_p = \begin{bmatrix} 2x_1 & -2y_1 & 3(x_2^2 - y_2^2) & -6x_2y_2 \\ 2y_1 & 2x_1 & 6x_2y_2 & 3(x_2^2 - y_2^2) \end{bmatrix}$$

- Now $S^3 = G^{-1}(0)$ where G is the smooth map

$$G: \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}.$$

$$(x_1, y_1, x_2, y_2) \mapsto x_1^2 + y_1^2 + x_2^2 + y_2^2.$$

So for $p = (x_1, y_1, x_2, y_2) \in G^{-1}(0)$ the tangent space

$$T_p S^3 = \text{kernel of } \left(G_* = DG_p = [2x_1, 2y_1, 2x_2, 2y_2] \right)$$

$$= \left\{ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in T_p(\mathbb{R}^4 \setminus \{0\}) \;\middle|\; x_1v_1 + y_1v_2 + x_2v_3 + y_2v_4 = 0 \right\}$$

- Since $T_p S^3$ is 3-dim'l, $T_p F^{-1}(0)$ is 2-dim'l and $T_p(\mathbb{R}^4 \setminus \{0\})$ is 4-dim'l, to prove (*) it suffices to find two linearly independent $v, w \in T_p S^3$ which are not in $T_p F^{-1}(0) = \text{kernel } DF_p$.

- Set $v = \begin{bmatrix} y_1 \\ -x_1 \\ 0 \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} 0 \\ 0 \\ y_2 \\ -x_2 \end{bmatrix}$

~~Both v, w are in $T_p S^3$ and if both $x_1, y_1 \neq 0$ and both $x_2, y_2 \neq 0$ they are linearly independent. Since we can't have $x_1 = y_1 = 0$ or $x_2 = y_2 = 0$, both v, w are nonzero elements of $T_p S^3$ which are linearly independent~~

For $p = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in S^3 \cap F^{-1}(0)$

we can't have $z_1 = 0$ or $z_2 = 0$.

Hence v, w are linearly independent vectors in $T_p S^3$, and.

$$DF_p(v) = \begin{bmatrix} 2x_1y_1 \\ 2(y_1^2 - x_1^2) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$DF_p(w) = \begin{bmatrix} 9x_2^2y_2 - 3y_2^3 \\ 9x_2y_2^2 - 3x_2^3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $v, w \notin$ kernel DF_p

2

3(c) By 3(b) $S^3 \cap F^{-1}(0)$ is a submfld of $\mathbb{R}^4 \setminus \{0\}$ of dim $3+2-4 = 1$.

Moreover, $S^3 \cap F^{-1}(0)$ is closed and hence compact and has no boundary.

By the classification of 1-dim mflds it must be a finite disjoint union of circles. (up to diffeomorphism)

To determine how many circles are in $S^3 \cap F^{-1}(0)$ lets consider the defining equations in polar coordinates $z_j = r_j e^{i\theta_j}$

$$\begin{cases} r_1^2 + r_2^2 = 1 & ① \\ r_1^2 e^{i2\theta_1} + r_2^3 e^{i3\theta_2} = 0 & ② \end{cases}$$

From ② we deduce that $r_1^2 = r_2^3$ ③

and $2\theta_1 = 3\theta_2 + \pi + 2\pi k$. ④

Together ①+③ $\Rightarrow$ $r_2^3 + r_2^2 = 1$. This determines r_2 and hence r_1 uniquely since there is one real solution to this equation, $r_2 \approx 3/4$.

From ④ we get

$$\theta_1 = \tfrac{3}{2}\theta_2 + \tfrac{\pi}{2} + \pi k$$

So $\theta_1 = \tfrac{3}{2}\theta_2 + \tfrac{\pi}{2} \mod 2\pi$ for k-even.

& $\theta_1 = \tfrac{3}{2}\theta_2 + \pi \mod 2\pi$ for k-odd.

In other words, θ_1 is a ~~t~~twice-valued function of θ_2

(Think $y = \pm\sqrt{1-x^2}$).

As θ_2 varies ~~[illegible]~~ one branch traces out the curve

$$\theta_2 \mapsto \left(r_1 e^{i(\frac{3}{2}\theta_2 + \frac{\pi}{2})}, r_2 e^{i\theta_2}\right)$$

and the other traces out the curve

$$\theta_2 \mapsto \left(r_1 e^{i(\frac{3}{2}\theta_2 + \pi)}, r_2 e^{i\theta_2}\right)$$

~~The end points~~ These curves are both closed with period 4π, and are disjoint. So $S^3 \cap F^{-1}(0)$ is diffeomorphic to a disjoint union of two circles, $O \amalg O$.