Problems

17-1. Show that every smooth compactly supported vector field is complete.

17-2. Compute the flow of each of the following vector fields on $\mathbb{R}^2$:

(a) $V = y\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$.

(b) $W = x\dfrac{\partial}{\partial x} + 2y\dfrac{\partial}{\partial y}$.

(c) $X = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$.

(d) $Y = x\dfrac{\partial}{\partial y} + y\dfrac{\partial}{\partial x}$.

17-3. For each of the vector fields in Problem 17-2, find smooth coordinates in a neighborhood of $(1,0)$ for which the given vector field is a coordinate vector field.

17-4. Let M be a connected smooth manifold. Show that the group of diffeomorphisms of M acts transitively on M. More precisely, for any two points $p, q \in M$, show that there is a diffeomorphism $F\colon M \to M$ such that $F(p) = q$. [Hint: First prove that if $p, q \in \mathbb{B}^n$ (the open unit ball in $\mathbb{R}^n$), there is a compactly supported smooth vector field on $\mathbb{B}^n$ whose flow θ satisfies $\theta_1(p) = q$.]

17-5. Let M be a smooth manifold. A curve $\gamma\colon \mathbb{R} \to M$ is *periodic* if there is a number $T > 0$ such that $\gamma(t) = \gamma(t + kT)$ for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. Suppose $X \in \mathfrak{X}(M)$ and γ is a maximal integral curve of X.

(a) Show that exactly one of the following holds:

- γ is constant.
- γ is injective.
- γ is periodic and nonconstant.

(b) If γ is periodic and nonconstant, show that there exists a unique positive number T (called the *period* of γ) such that $\gamma(t) = \gamma(t')$ if and only if $t - t' = kT$ for some $k \in \mathbb{Z}$.

(c) Show that the image of γ is an immersed submanifold of M, diffeomorphic to $\mathbb{R}$, $\mathbb{S}^1$, or $\mathbb{R}^0$.

17-6. Let M be a smooth n-manifold, and suppose V is a smooth vector field on M such that every integral curve of V is periodic with the same period (see Problem 17-5). Define an equivalence relation on M by saying $p \sim q$ if p and q are in the image of the same integral curve of V. Let $M/{\sim}$ be the quotient space, and let $\pi\colon M \to M/{\sim}$ be the quotient map. Show that $M/{\sim}$ is a topological $(n-1)$-manifold and has a unique smooth structure such that π is a submersion.

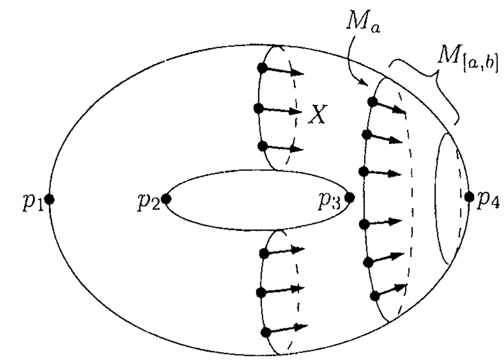

Figure 17.10. The setup for Problem 17-10.

17-7. Let M be a connected smooth 1-manifold. Show that M is diffeomorphic to either $\mathbb{R}$ or $\mathbb{S}^1$, as follows:

(a) First do the case in which M is orientable by showing that M admits a nonvanishing smooth vector field and using Problem 17-5.

(b) If M is arbitrary, prove that M is orientable by showing that its universal covering manifold is diffeomorphic to $\mathbb{R}$ and that every orientation-reversing diffeomorphism $\varphi\colon \mathbb{R} \to \mathbb{R}$ has a fixed point.

Conclude that the the smooth structures on $\mathbb{R}$ and $\mathbb{S}^1$ are unique up to diffeomorphism.

17-8. Let θ be a smooth flow on an oriented smooth manifold. Show that for each $t \in \mathbb{R}$, θ_t is orientation-preserving wherever it is defined.

17-9. Let M be a smooth manifold, and let $S \subset M$ be a compact embedded hypersurface. Suppose $N \in \mathfrak{X}(M)$ is a smooth vector field that is transverse to S. Show that for some $\varepsilon > 0$, the flow of N restricts to a diffeomorphism from $(-\varepsilon, \varepsilon) \times S$ to a neighborhood of S in M.

17-10. Let M be a compact Riemannian n-manifold, and let $f \in C^\infty(M)$. Suppose f has only finitely many critical points $\{p_1, \dots, p_k\}$ with corresponding critical values $\{c_1, \dots, c_k\}$. (Assume without loss of generality that $c_1 \le \dots \le c_k$.) For any $a < b \in \mathbb{R}$, define $M_a = f^{-1}(a)$, $M_{[a,b]} = f^{-1}([a,b])$, and $M_{(a,b)} = f^{-1}((a,b))$. If a is a regular value, note that M_a is an embedded hypersurface in M (see Figure 17.10).

(a) Let X be the vector field $X = \operatorname{grad} f / |\operatorname{grad} f|_g^2$ on $M \smallsetminus \{p_1, \dots, p_k\}$, and let θ denote the flow of X. Show that $f(\theta_t(p)) = f(p) + t$ whenever $\theta_t(p)$ is defined.

(b) Let $[a, b] \subset \mathbb{R}$ be a compact interval containing no critical values of f. Show that

$$\theta : [0, b-a] \times M_a \to M_{[a,b]}$$

is a diffeomorphism.

[Remark: This result shows that M can be decomposed as a union of simpler "building blocks"—the product sets $M_{[c_i+\varepsilon, c_{i+1}-\varepsilon]} \approx I \times M_{c_i+\varepsilon}$, and the neighborhoods $M_{(c_i-\varepsilon, c_i+\varepsilon)}$ of the critical points. This is the starting point of *Morse theory*, which is one of the deepest applications of differential geometry to topology. The next step would be to analyze the behavior of f near each critical point, and use this analysis to determine exactly how the level sets change topologically when crossing a critical level. See [Mil63] for an excellent introduction.]

17-11. Suppose M is a smooth manifold that admits a proper smooth function $f : M \to \mathbb{R}$ with no critical points. Show that M is diffeomorphic to $N \times \mathbb{R}$ for some compact smooth manifold N. [Hint: Let $X = \operatorname{grad} f / |\operatorname{grad} f|_g^2$, defined with respect to some Riemannian metric on M. Show that X is complete, and use its flow to define the diffeomorphism. See also Problems 17-9 and 17-10.]

17-12. Let S be an embedded hypersurface in a smooth manifold M, and let N be a smooth vector field on M that is transverse to S.

(a) For any $f \in C^\infty(M)$ and $\varphi \in C^\infty(S)$, show that there is a neighborhood U of S in M and a unique function $u \in C^\infty(U)$ that satisfies

$$Nu = f,$$
$$u|_S = \varphi.$$

[Hint: First consider the case $M = \mathbb{R} \times S$ and $N = \partial/\partial t$, and then use the flow of N to reduce to this case locally.]

(b) Use the method of part (a) to find an explicit solution $u(x, y)$ to the following problem:

$$y\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x,$$
$$u(x, 0) = \sin x.$$

[Remark: An equation of the form $Nu = f$ is called a *first-order linear partial differential equation* for u; first-order because it involves only first derivatives of u, and linear because the left-hand side depends linearly on u. This problem shows that solving such a PDE reduces to solving a system of ODEs (namely, the system one has to solve to find the flow of N).]

17-13. Let M be a smooth manifold with boundary. A subset $C \subset M$ containing ∂M is called a *collar* if C is the image of a smooth embedding $[0, 1] \times \partial M \to M$ that restricts to the obvious identification $\{0\} \times \partial M \to \partial M$. This problem shows that every smooth manifold with boundary has a collar.

(a) Show that there exists a vector field $N \in \mathfrak{X}(M)$ whose restriction to ∂M is inward-pointing.

(b) For any positive real-valued function $\delta : \partial M \to \mathbb{R}$, define a subset $\mathcal{D}_\delta \subset [0, \infty) \times \partial M$ by

$$\mathcal{D}_\delta = \{(t, p) : p \in \partial M,\ 0 \le t \le \delta(p)\}.$$

Show that there are a smooth positive function $\delta : \partial M \to \mathbb{R}$ and a smooth map $\theta : \mathcal{D}_\delta \to M$ such that for each $p \in \partial M$, the map $t \mapsto \theta(t, p)$ is an integral curve of N starting at p. [Hint: Use the ODE theorem in local coordinates around each point of ∂M, and define δ by means of a partition of unity.]

(c) Show that θ is an embedding.

(d) Show that the image of θ is a collar.

17-14. Suppose M is a smooth manifold with boundary.

(a) Show that the inclusion $\operatorname{Int} M \hookrightarrow M$ is a homotopy equivalence.

(b) Show that there is a smooth manifold without boundary $\widetilde{M}$ and a smooth embedding $M \hookrightarrow \widetilde{M}$ that is also a homotopy equivalence.

(c) The *double* of M is the quotient space $D(M)$ obtained from $M \amalg M$ by identifying each point in the boundary of the first copy of M with the corresponding point in the boundary of the second copy. Show that $D(M)$ is a topological n-manifold (without boundary) and that it has a smooth structure such that M is a smoothly embedded submanifold with boundary.

[Hint: Use a collar.]

17-15. Suppose M is a smooth manifold, $J \subset \mathbb{R}$ is an open interval, and $V : J \times M \to TM$ is a smooth time-dependent vector field on M. Let $\theta : \mathcal{E} \to M$ be the time-dependent flow of V. For any $(t, s) \in J \times J$, let $M_{t,s}$ denote the set $\{p \in M : (t, s, p) \in \mathcal{E}\}$, and define $\theta_{t,s} : M_{t,s} \to M$ by $\theta_{t,s}(p) = \theta(t, s, p)$.

(a) If $(t_1, t_0, p) \in \mathcal{E}$ and $(t_2, t_1, \theta_{t_1,t_0}(p)) \in \mathcal{E}$, show that $(t_2, t_0, p) \in \mathcal{E}$ and

$$\theta_{t_2,t_1} \circ \theta_{t_1,t_0}(p) = \theta_{t_2,t_0}(p).$$

(b) For any $(t, s) \in J \times J$, show that $M_{t,s}$ is open in M, and $\theta_{t,s} : M_{t,s} \to M_{s,t}$ is a diffeomorphism with inverse $\theta_{s,t}$.

(c) If M is compact, show that $\mathcal{E} = J \times J \times M$.

If (M,ω,H) is a Hamiltonian system, any function $f \in C^\infty(M)$ that is constant on every integral curve of X_H is called a *conserved quantity* of the system. Conserved quantities turn out to be deeply related to symmetries, as we now show.

A smooth vector field V on M is called an *infinitesimal symmetry* of (M,ω,H) if both ω and H are invariant under the flow of V.

◇ **Exercise 18.6.** Let (M,ω,H) be a Hamiltonian system.

(a) Show that $f \in C^\infty(M)$ is a conserved quantity if and only if $\{H,f\} = 0$.
(b) Show that the infinitesimal symmetries are precisely the symplectic vector fields V that satisfy $VH = 0$.
(c) If θ is the flow of an infinitesimal symmetry and γ is a trajectory of the system, show that for each $s \in \mathbb{R}$, $\theta_s \circ \gamma$ is also a trajectory on its domain of definition.

The following theorem, first proved (in a somewhat different form) by Emmy Noether in 1918 [Noe71], has had a profound influence on both physics and mathematics. It shows that for many manifolds (simply connected ones, for example) there is a one-to-one correspondence between conserved quantities (modulo constants) and infinitesimal symmetries.

Theorem 18.27 (Noether's Theorem). *Let (M,ω,H) be a Hamiltonian system. If f is any conserved quantity, then its Hamiltonian vector field is an infinitesimal symmetry. Conversely, if $H^1_{dR}(M) = 0$, then every infinitesimal symmetry is the Hamiltonian vector field of a conserved quantity, which is unique up to addition of a function that is constant on each component of M.*

Proof. Suppose f is a conserved quantity. Exercise 18.6 shows that $\{H,f\} = 0$. This in turn implies that $X_f H = \{H,f\} = 0$, so H is constant along the flow of X_f. Since ω is invariant along the flow of any Hamiltonian vector field by Proposition 18.23, this shows that X_f is an infinitesimal symmetry.

Now suppose that $H^1_{dR}(M) = 0$ and let V be an infinitesimal symmetry. Then V is symplectic by Exercise 18.6, and globally Hamiltonian by Proposition 18.23. Writing $V = X_f$, the fact that H is constant along the flow of V implies that $\{H,f\} = X_f H = VH = 0$, so f is a conserved quantity. If $\widetilde{f}$ is any other function that satisfies $X_{\widetilde{f}} = V = X_f$, then $d(\widetilde{f} - f) = (X_{\widetilde{f}} - X_f) \lrcorner \omega = 0$, so $\widetilde{f} - f$ must be constant on each component of M. □

There is one conserved quantity that every Hamiltonian system possesses: the Hamiltonian H itself. The infinitesimal symmetry corresponding to it, of course, generates the Hamiltonian flow of the system, which describes how the system evolves over time. Since H is typically interpreted as the total energy of the system (as in Example 18.24), one usually says that the symmetry corresponding to conservation of energy is "translation in the time variable."

Problems

18-1. Give an example of smooth vector fields V, $\widetilde{V}$, and W on $\mathbb{R}^2$ such that $V = \widetilde{V} = \partial/\partial x$ along the x-axis but $\mathcal{L}_V W \neq \mathcal{L}_{\widetilde{V}} W$ at the origin. [Remark: This shows that it is really necessary to know the vector field V to compute $(\mathcal{L}_V W)_p$; it is not sufficient just to know the vector V_p, or even to know the values of V along an integral curve of V.]

18-2. For each k-tuple of vector fields on $\mathbb{R}^3$ shown below, either find smooth coordinates (u^1,u^2,u^3) in a neighborhood of $(1,0,0)$ such that $V_i = \partial/\partial u^i$ for $i = 1,\dots,k$, or explain why there are none.

(a) $k = 2$; $V_1 = \dfrac{\partial}{\partial x}$, $V_2 = \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$.

(b) $k = 2$; $V_1 = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$, $V_2 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$.

(c) $k = 3$; $V_1 = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$, $V_2 = y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y}$, $V_3 = z\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial z}$.

18-3. This problem generalizes the result of Problem 17-12. Let M be a smooth n-manifold, and let $S \subset M$ be a smooth embedded submanifold of codimension k. Suppose $X_1,\dots,X_k$ are commuting independent smooth vector fields on M whose span is complementary to T_pS at each $p \in S$. If $f_1,\dots,f_k \in C^\infty(M)$ are functions such that $X_i f_j = X_j f_i$ for all $i,j = 1,\dots,k$, and $\varphi \in C^\infty(S)$ is arbitrary, show that there exist a neighborhood U of S in M and a unique function $u \in C^\infty(U)$ satisfying

$$X_i u = f_i, \qquad i = 1,\dots,k;$$
$$u|_S = \varphi.$$

18-4. Let M,N be smooth manifolds, and suppose $\pi\colon M \to N$ is a surjective submersion with connected fibers. We say that a tangent vector $X \in T_pM$ is *vertical* if $\pi_* X = 0$. Suppose $\omega \in \mathcal{A}^k(M)$. Show that there exists $\eta \in \mathcal{A}^k(N)$ such that $\omega = \pi^*\eta$ if and only if $X \lrcorner \omega_p = 0$ and $X \lrcorner d\omega_p = 0$ for every $p \in M$ and every vertical vector $X \in T_pM$. [Hint: First do the case in which $\pi\colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ is projection onto the first n coordinates.]

18-5. Define vector fields V and W on the plane by

$$V = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}; \quad W = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Compute the flows θ, ψ of V and W, and verify that they do not commute by finding explicit times s and t such that $\theta_s \circ \psi_t \neq \psi_t \circ \theta_s$.

18-6. Let M be a smooth manifold and $X \in \mathfrak{T}(M)$. Show that the Lie derivative operators on covariant tensor fields, $\mathcal{L}_X : \mathfrak{T}^k(M) \to \mathfrak{T}^k(M)$ for $k \geq 0$, are uniquely characterized by the following properties:

(a) $\mathcal{L}_X f = Xf$ for $X \in \mathfrak{T}^0(M) = C^\infty(M)$.
(b) $\mathcal{L}_X(\sigma \otimes \tau) = \mathcal{L}_X \sigma \otimes \tau + \sigma \otimes \mathcal{L}_X \tau$ for $\sigma \in \mathfrak{T}^k(M)$, $\tau \in \mathfrak{T}^l(M)$.
(c) $\mathcal{L}_X(\omega(Y)) = \mathcal{L}_X \omega(Y) + \omega(\mathcal{L}_X Y)$ for $\omega \in \mathfrak{T}^1(M)$, $Y \in \mathfrak{T}(M)$.

[Remark: The Lie derivative operators on tensor fields are sometimes *defined* as the unique operators satisfying these properties.]

18-7. Let (M, ω) be a symplectic manifold.

(a) Show that the set of symplectic vector fields on M is a Lie subalgebra of $\mathfrak{T}(M)$.
(b) Show that the set of Hamiltonian vector fields is a Lie subalgebra of the set of symplectic vector fields.
(c) Show that the quotient of the symplectic vector fields modulo the Hamiltonian vector fields is isomorphic (as a vector space) to $H^1_{dR}(M)$.

18-8. Using the same technique as in the proof of the Darboux theorem, prove the following theorem of Moser: If M is an oriented compact smooth manifold, and Ω_0, Ω_1 are smooth orientation forms on M such that $\int_M \Omega_0 = \int_M \Omega_1$, then there exists a diffeomorphism $F : M \to M$ such that $F^*\Omega_1 = \Omega_0$.

18-9. Prove the following global version of the Darboux theorem: Suppose M is a compact symplectic manifold, and ω_0, ω_1 are cohomologous symplectic forms on M. Show that there is a diffeomorphism $F : M \to M$ such that $F^*\omega_1 = \omega_0$.

18-10. This problem outlines a different proof of the Darboux theorem. Let (M, ω) be a $2n$-dimensional symplectic manifold and $p \in M$.

(a) Show that smooth coordinates (x^i, y^i) on an open set $U \subset M$ are Darboux coordinates if and only if their Poisson brackets satisfy

$$\{x^i, y^j\} = \delta^{ij}; \qquad \{x^i, x^j\} = \{y^i, y^j\} = 0. \tag{18.25}$$

(b) Prove by induction on k that for each $k = 0, \dots, n$, there are functions $(x^1, y^1, \dots, x^k, y^k)$ satisfying (18.25) near p such that $\{dx^1, dy^1, \dots, dx^k, dy^k\}$ are independent at p. When $k = n$, this proves the theorem. [Hint: For the inductive step, assuming that $(x^1, y^1, \dots, x^k, y^k)$ have been found, find smooth coordinates $(u^1, \dots, u^{2n})$ such that

$$\frac{\partial}{\partial u^i} = X_{x^i}, \qquad \frac{\partial}{\partial u^{i+k}} = X_{y^i}, \qquad i = 1, \dots, k,$$

and let $y^{k+1} = u^{2k+1}$. Then find new coordinates $(v^1, \dots, v^{2n})$ such that

$$\frac{\partial}{\partial v^i} = X_{x^i}, \qquad i = 1, \dots, k,$$
$$\frac{\partial}{\partial v^{i+k}} = X_{y^i}, \qquad i = 1, \dots, k+1,$$

and let $x^{k+1} = v^{2k+1}$.]

18-11. Consider the 2-body problem in $\mathbb{R}^3$, i.e., the Hamiltonian system (T^*Q, ω, E) described in Example 18.24 in the special case $n = 2$. Suppose that the potential energy V depends only on the distance between the particles. More precisely, suppose that $V(q) = v(r(q))$ for some smooth function $v : (0, \infty) \to \mathbb{R}$, where

$$r(q) = \sqrt{(q^1 - q^4)^2 + (q^2 - q^5)^2 + (q^3 - q^6)^2}.$$

(a) Show that the function $f : T^*Q \to \mathbb{R}$ defined by

$$f(p, q) = p^1 + p^4$$

is a conserved quantity (called the linear momentum in the x-direction), and that the corresponding infinitesimal symmetry generates translations in the x-direction:

$$\theta_t(q, p) = \left(q^1 + t, q^2, q^3, q^4 + t, q^5, q^6, p_1, \dots, p_6\right).$$

(b) Show that the function $\alpha : T^*Q \to \mathbb{R}$ defined by

$$\alpha(p, q) = q^1 p_2 - q^2 p_1 + q^4 p_5 - q^5 p_4$$

is a conserved quantity (called the angular momentum about the z-axis), and find the flow of the corresponding infinitesimal symmetry. Explain what this has to do with rotational symmetry.