and, when $\dim M = 3$,

$$\operatorname{curl} X = \left(*\, dX^\flat\right)^\#.$$

14-16. Let (M,g) be a compact, oriented Riemannian n-manifold. For $1 \le k \le n$, define a map $d^*\colon \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$ by $d^*\omega = (-1)^{n(k+1)+1} * d * \omega$, where $*$ is the Hodge star operator defined in Problem 14-12. Extend this definition to 0-forms by defining $d^*\omega = 0$ for $\omega \in \mathcal{A}^0(M)$.

(a) Show that $d^* \circ d^* = 0$.

(b) Show that the formula

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle_g \, dV_g$$

defines an inner product on $\mathcal{A}^k(M)$ for each k, where $\langle \cdot, \cdot \rangle_g$ is the pointwise inner product on forms defined in Problem 14-12.

(c) Show that $(d^*\omega, \eta) = (\omega, d\eta)$ for all $\omega \in \mathcal{A}^k(M)$ and $\eta \in \mathcal{A}^{k-1}(M)$.

14-17. On $\mathbb{R}^3$ with the Euclidean metric, show that the curl operator we have defined is given by the classical formula:

$$\begin{aligned}\operatorname{curl}&\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}\right)\\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial}{\partial x} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\frac{\partial}{\partial z}.\end{aligned}$$

14-18. Show that any finite product $M_1 \times \cdots \times M_k$ of smooth manifolds with corners is again a smooth manifold with corners. Give a counterexample to show that a finite product of smooth manifolds with boundary need not be a smooth manifold with boundary.

14-19. Suppose M is a smooth manifold with corners, and let $\mathcal{C}$ denote the set of corner points of M. Show that $M \smallsetminus \mathcal{C}$ is a smooth manifold with boundary.

14-20. Show that the divergence operator on an oriented Riemannian manifold does not depend on the choice of orientation, and conclude that it is invariantly defined on all Riemannian manifolds.

14-21. Let M and N be compact, connected, oriented, smooth manifolds, and suppose $F, G\colon M \to N$ are diffeomorphisms. If F and G are homotopic, show that they are either both orientation-preserving or both orientation-reversing. [Hint: Use the Whitney approximation theorem and Stokes's theorem on $M \times I$.]

14-22. The Hairy Ball Theorem: *There exists a nowhere-vanishing vector field on $\mathbb{S}^n$ if and only if n is odd.* ("You cannot comb the hair on a ball.") Prove this by showing that the following are equivalent:

(a) There exists a nowhere-vanishing vector field on $\mathbb{S}^n$.

(b) There exists a continuous map $V\colon \mathbb{S}^n \to \mathbb{S}^n$ satisfying $V(x) \perp x$ (with respect to the Euclidean dot product on $\mathbb{R}^{n+1}$) for all $x \in \mathbb{S}^n$.

(c) The antipodal map $\alpha\colon \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to $\mathrm{Id}_{\mathbb{S}^n}$.

(d) The antipodal map $\alpha\colon \mathbb{S}^n \to \mathbb{S}^n$ is orientation-preserving.

(e) n is odd.

[Hint: Use Problems 8-7, 13-5, and 14-21.]

and, when $\dim M = 3$,

$$\operatorname{curl} X = \left(* \, dX^\flat\right)^\#.$$

14-16. Let (M,g) be a compact, oriented Riemannian n-manifold. For $1 \le k \le n$, define a map $d^*\colon \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$ by $d^*\omega = (-1)^{n(k+1)+1} * d * \omega$, where $*$ is the Hodge star operator defined in Problem 14-12. Extend this definition to 0-forms by defining $d^*\omega = 0$ for $\omega \in \mathcal{A}^0(M)$.

(a) Show that $d^* \circ d^* = 0$.

(b) Show that the formula

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle_g \, dV_g$$

defines an inner product on $\mathcal{A}^k(M)$ for each k, where $\langle \cdot, \cdot \rangle_g$ is the pointwise inner product on forms defined in Problem 14-12.

(c) Show that $(d^*\omega, \eta) = (\omega, d\eta)$ for all $\omega \in \mathcal{A}^k(M)$ and $\eta \in \mathcal{A}^{k-1}(M)$.

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14-22. THE HAIRY BALL THEOREM: *There exists a nowhere-vanishing vector field on* $\mathbb{S}^n$ *if and only if* n *is odd.* ("You cannot comb the hair on a ball.") Prove this by showing that the following are equivalent:

(a) There exists a nowhere-vanishing vector field on $\mathbb{S}^n$.

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(c) The antipodal map $\alpha\colon \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to $\mathrm{Id}_{\mathbb{S}^n}$.

(d) The antipodal map $\alpha\colon \mathbb{S}^n \to \mathbb{S}^n$ is orientation-preserving.

(e) n is odd.

[Hint: Use Problems 8-7, 13-5, and 14-21.]

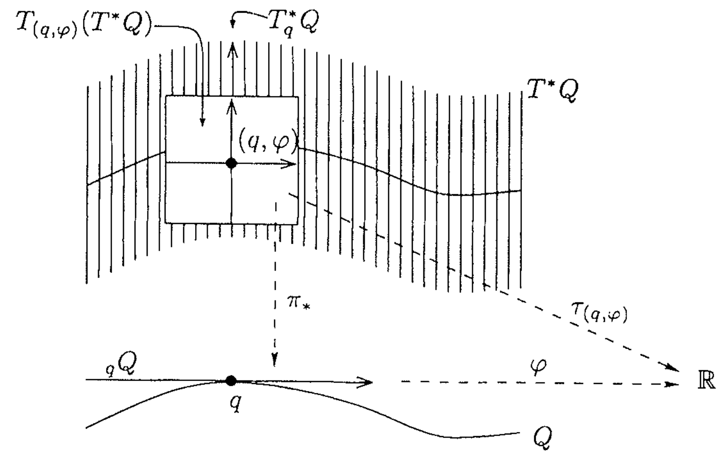

Figure 12.4. The tautological 1-form on T^*Q.

Q has the coordinate expression $\pi(x,\xi) = x$, and therefore the coordinate representation of τ is

$$\tau_{(x,\xi)} = \pi^*\left(\xi_i\, dx^i\right) = \xi_i\, dx^i.$$

It follows immediately that τ is smooth, because its component functions are linear.

Clearly, ω is closed, because it is exact. Moreover,

$$\omega = -d\tau = \sum_i dx^i \wedge d\xi_i.$$

Under the identification of an open subset of T^*Q with an open subset of $\mathbb{R}^{2n}$ by means of these coordinates, ω corresponds to the standard symplectic form on $\mathbb{R}^{2n}$ (with ξ_i substituted for y^i). It follows that ω is symplectic. □

The symplectic form defined in this proposition is called the *canonical symplectic form* on T^*Q. One of its many uses is in giving a somewhat more "geometric" interpretation of what it means for a 1-form to be closed, as shown by the following proposition.

Proposition 12.25. *Let M be a smooth manifold, and let σ be a smooth 1-form on M. Thought of as a smooth map from M to T^*M, σ is a smooth embedding, and σ is closed if and only if its image $\sigma(M)$ is a Lagrangian submanifold of T^*M.*

Proof. Throughout this proof we need to remember that σ is playing two roles: On the one hand, it is a 1-form on M, and on the other hand, it is a smooth map between manifolds. Since they are literally the same map, we will not use different notations to distinguish between them; but you should be careful to think about which role σ is playing at each step of the argument.

In terms of any smooth local coordinates (x^i) for M and the corresponding standard coordinates (x^i, ξ_i) for T^*M, the map $\sigma\colon M \to T^*M$ has the coordinate representation

$$\sigma\left(x^1,\dots,x^n\right) = \left(x^1,\dots,x^n,\sigma_1(x),\dots,\sigma_n(x)\right),$$

where $\sigma_i\, dx^i$ is the coordinate representation of σ as a 1-form. It follows immediately that σ is an immersion, and the fact that it is injective follows from $\pi \circ \sigma = \mathrm{Id}_M$.

To show that it is an embedding, it suffices by Proposition 7.4 to show that it is a proper map. This in turn follows from the fact that π is a left inverse for σ, by Lemma 2.16.

Because $\sigma(M)$ is n-dimensional, it is Lagrangian if and only if it is isotropic, which is the case if and only if $\sigma^*\omega = 0$. The pullback of the tautological form τ under σ is

$$\sigma^*\tau = \sigma^*\left(\xi_i\, dx^i\right) = \sigma_i\, dx^i = \sigma.$$

This can also be seen somewhat more invariantly from the computation

$$(\sigma^*\tau)_p(X) = \tau_{\sigma(p)}(\sigma_* X) = \sigma_p(\pi_*\sigma_* X) = \sigma_p(X),$$

which follows from the definition of τ and the fact that $\pi \circ \sigma = \mathrm{Id}_M$. Therefore,

$$\sigma^*\omega = -\sigma^* d\tau = -d(\sigma^*\tau) = -d\sigma.$$

It follows that σ is a Lagrangian embedding if and only if $d\sigma = 0$. □

Problems

12-1. Let $v_1,\dots,v_n$ be any n vectors in $\mathbb{R}^n$, and let P be the n-dimensional parallelepiped spanned by them:

$$P = \{t_1 v_1 + \cdots + t_n v_n : 0 \le t_i \le 1\}.$$

Show that $\mathrm{Vol}(P) = |\det(v_1,\dots,v_n)|$.

12-2. Let $\left(e^1, e^2, e^3\right)$ be the standard dual basis for $\left(\mathbb{R}^3\right)^*$. Show that $e^1 \otimes e^2 \otimes e^3$ is not equal to a sum of an alternating tensor and a symmetric tensor.

12-3. Show that covectors $\omega^1,\dots,\omega^k$ on a finite-dimensional vector space are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.

12-4. Show that two k-tuples $\{\omega^1,\dots,\omega^k\}$ and $\{\eta^1,\dots,\eta^k\}$ of independent covectors have the same span if and only if

$$\omega^1 \wedge \cdots \wedge \omega^k = c\,\eta^1 \wedge \cdots \wedge \eta^k$$

for some nonzero real number c.

12-5. A k-covector η on a finite-dimensional vector space V is said to be *decomposable* if it can be written

$$\eta = \omega^1 \wedge \cdots \wedge \omega^k,$$

where $\omega^1, \dots, \omega^k$ are covectors. For what values of n is it true that every 2-covector on $\mathbb{R}^n$ is decomposable?

12-6. Define a 2-form Ω on $\mathbb{R}^3$ by

$$\Omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy.$$

(a) Compute Ω in spherical coordinates (ρ, φ, θ) defined by $(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)$.
(b) Compute $d\Omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.
(c) Compute the restriction $\Omega|_{\mathbb{S}^2} = \iota^*\Omega$, using coordinates (φ, θ), on the open subset where these coordinates are defined.
(d) Show that $\Omega|_{\mathbb{S}^2}$ is nowhere zero.

12-7. In each of the following problems, $g\colon M \to N$ is a smooth map between manifolds M and N, and ω is a smooth differential form on N. In each case, compute $g^*\omega$ and $d\omega$, and verify by direct computation that $g^*(d\omega) = d(g^*\omega)$.

(a) $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$(x, y) = g(s, t) = (st, e^t);$$
$$\omega = x\,dy.$$

(b) $g\colon \mathbb{R}^2 \to \mathbb{R}^3$ by

$$(x, y, z) = g(\theta, \varphi) = ((\cos\varphi + 2)\cos\theta, (\cos\varphi + 2)\sin\theta, \sin\varphi);$$
$$\omega = y\,dz \wedge dx.$$

(c) $g\colon \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\} \to \mathbb{R}^3 \smallsetminus \{0\}$ by

$$(x, y, z) = g(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right);$$
$$\omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

12-8. Let V be a finite-dimensional real vector space. We have two ways to think about the tensor space $T^k(V)$: concretely, as the space of k-multilinear functionals on V; and abstractly, as the tensor product space $V^* \otimes \cdots \otimes V^*$. However, we have defined alternating and symmetric tensors only in terms of the concrete definition. This problem outlines an abstract approach to alternating tensors. (Symmetric tensors can be handled similarly.)

Let $\mathcal{A}$ denote the subspace of $V^* \otimes \cdots \otimes V^*$ spanned by all elements of the form $\alpha \otimes \varphi \otimes \varphi \otimes \beta$ for a covector φ and arbitrary tensors α, β, and let $A^k(V^*)$ denote the quotient vector space $(V^* \otimes \cdots \otimes V^*)/\mathcal{A}$. Show that there is a unique isomorphism $F\colon A^k(V^*) \to \Lambda^k(V)$ such that the following diagram commutes:

$$\begin{array}{ccc} V^* \otimes \cdots \otimes V^* & \xrightarrow{\ \cong\ } & T^k(V) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{Alt}} \\ A^k(V^*) & \xrightarrow[F]{} & \Lambda^k(V). \end{array}$$

Define a wedge product on $A^k(V^*)$ by $\omega \wedge \eta = \pi(\widetilde{\omega} \otimes \widetilde{\eta})$, where $\pi\colon V^* \otimes \cdots \otimes V^* \to A^k(V^*)$ is the projection, and $\widetilde{\omega}$, $\widetilde{\eta}$ are arbitrary tensors such that $\pi(\widetilde{\omega}) = \omega$, $\pi(\widetilde{\eta}) = \eta$. Show that this wedge product is well-defined, and that F takes this wedge product to the Alt convention wedge product on $\Lambda^k(V)$. [Remark: This is one reason why some authors consider the Alt convention for the wedge product to be more natural than the determinant convention. It also explains why some authors prefer the notation $\Lambda^k(V^*)$ instead of $\Lambda^k(V)$ for the space of alternating covariant k-tensors, since it can be viewed as a quotient of the k-fold tensor product of V^* with itself.]

12-9. Let (V, ω) be a symplectic vector space of dimension $2n$. Show that for each symplectic, isotropic, coisotropic, or Lagrangian subspace $S \subset V$, there exists a symplectic basis (A_i, B_i) for V with the following property:

(a) If S is symplectic, $S = \operatorname{span}(A_1, B_1, \dots, A_k, B_k)$ for some k.
(b) If S is isotropic, $S = \operatorname{span}(A_1, \dots, A_k)$ for some k.
(c) If S is coisotropic, $S = \operatorname{span}(A_1, \dots, A_n, B_1, \dots, B_k)$ for some k.
(d) If S is Lagrangian, $S = \operatorname{span}(A_1, \dots, A_n)$.

12-10. Let (M, ω) be a symplectic manifold, and suppose $F\colon N \to M$ is a smooth map such that $F^*\omega$ is symplectic. Show that F is an immersion.

12-11. Let Q be a smooth manifold, and let S be an embedded submanifold of the total space of T^*Q. Show that S is the image of a smooth closed 1-form on Q if and only if S is Lagrangian, transverse to the fibers, and intersects each fiber in exactly one point. (Two submanifolds N_1, N_2 of a smooth manifold M are said to be *transverse* if $T_pN_1 + T_pN_2$ spans T_pM at each point $p \in N_1 \cap N_2$. See also Problem 8-17.)

Proposition 13.26. *Suppose M is any Riemannian manifold with boundary. There is a unique smooth outward-pointing unit normal vector field N along ∂M.*

Proof. First we prove uniqueness. At any point $p \in \partial M$, the vector space $(T_p\partial M)^\perp \subset T_pM$ is 1-dimensional, so there are exactly two unit vectors at p that are normal to ∂M. Since any unit normal vector N is obviously transverse to ∂M, it must have nonzero x^n-component in any smooth boundary chart. Thus exactly one of the two choices of unit normal has negative x^n-component, which is equivalent to being outward-pointing.

To prove existence, we will show that there exists a smooth outward unit normal field in a neighborhood of each point. By the uniqueness result above, these vector fields all agree where they overlap, so the resulting vector field is globally defined.

Let $p \in \partial M$. By Proposition 11.24, there exists a smooth adapted orthonormal frame $(E_1, \dots, E_n)$ in a neighborhood U of p. In this frame, E_n is a smooth unit normal vector field along ∂M. If we assume (by shrinking U if necessary) that U is connected, then E_n must be either inward-pointing or outward-pointing on all of $\partial M \cap U$. Replacing E_n by $-E_n$ if necessary, we obtain a smooth outward-pointing unit normal vector field defined near p. This completes the proof. □

The next corollary is immediate.

Corollary 13.27. *If (M, g) is an oriented Riemannian manifold with boundary and $\widetilde{g}$ is the induced Riemannian metric on ∂M, then the volume form of $\widetilde{g}$ is*

$$dV_{\widetilde{g}} = (N \lrcorner dV_g)|_{\partial M},$$

where N is the outward unit normal vector field along ∂M.

Problems

13-1. Suppose M is a smooth manifold that is the union of two orientable open submanifolds with connected intersection. Show that M is orientable. Use this to give another proof that $\mathbb{S}^n$ is orientable.

13-2. Suppose $\pi\colon \widetilde{M} \to M$ is a smooth covering map and M is orientable. Show that $\widetilde{M}$ is also orientable.

13-3. Suppose M and N are oriented smooth manifolds and $F\colon M \to N$ is a local diffeomorphism. If M is connected, show that F is either orientation-preserving or orientation-reversing.

13-4. Suppose M is a connected, oriented, smooth manifold and Γ is a discrete group acting smoothly, freely, and properly on M. We say that the action is orientation-preserving if for each $\gamma \in \Gamma$, the diffeomorphism $x \mapsto \gamma \cdot x$ is orientation-preserving. Show that M/Γ is orientable if and only if the action of Γ is orientation-preserving.

13-5. Let $\alpha\colon \mathbb{S}^n \to \mathbb{S}^n$ be the antipodal map: $\alpha(x) = -x$. Show that α is orientation-preserving if and only if n is odd.

13-6. Prove that $\mathbb{RP}^n$ is orientable if and only if n is odd.

13-7. If ω is a symplectic form on a $2n$-manifold, show that $\omega \wedge \cdots \wedge \omega$ (the n-fold wedge product of ω with itself) is a nonvanishing $2n$-form on M, and thus every symplectic manifold is orientable.

13-8. Suppose M is an oriented Riemannian manifold, and $S \subset M$ is an oriented hypersurface (with or without boundary). Show that there is a unique smooth unit normal vector field along S that determines the given orientation of S.

13-9. Suppose M is a smooth orientable Riemannian manifold and $S \subset M$ is an immersed or embedded submanifold.

(a) If S has trivial normal bundle (see page 282), show that S is orientable.

(b) If S is an orientable hypersurface, show that S has trivial normal bundle.

13-10. Let M be a connected, nonorientable smooth manifold, and let $\widehat{\pi}\colon \widehat{M} \to M$ be its orientation covering.

(a) If $\widetilde{M}$ is an orientable smooth manifold and $\pi\colon \widetilde{M} \to M$ is a smooth covering map, show that there exists a smooth map $\varphi\colon \widetilde{M} \to \widehat{M}$ such that $\widehat{\pi} \circ \varphi = \pi$. [Hint: First define a smooth map $\widetilde{\varphi}\colon \widetilde{M} \to \Lambda^n_* M$ by setting $\widetilde{\varphi}(p) = \sigma^*\Omega_p$ locally, where Ω is an orientation form for $\widetilde{M}$ and σ is a suitable local section of π.]

(b) Uniqueness of the Orientation Covering: If $\pi\colon \widetilde{M} \to M$ is as above and in addition π is a two-sheeted covering, show that φ is a diffeomorphism.

13-11. Suppose S is an oriented embedded 2-manifold with boundary in $\mathbb{R}^3$, and let $C = \partial S$ with the induced orientation. By Problem 13-8, there is a unique smooth unit normal vector field N on S that determines the orientation. Let T be the oriented unit tangent vector field on C, and let V be the unique unit vector field tangent to S along C that is orthogonal to T and inward-pointing. Show that (T_p, V_p, N_p) is an oriented orthonormal basis for $\mathbb{R}^3$ at each $p \in C$.

13-12. Let E be the total space of the Möbius bundle, which is the quotient of $\mathbb{R}^2$ by the $\mathbb{Z}$-action $n \cdot (x, y) = (x + n, (-1)^n y)$ (see Problem 9-18). The *Möbius band* is the subset $M \subset E$ that is the image under the

Using these facts, together with the divergence theorem on $\widehat{M}$ and the result of Problem 14-5, we compute

$$\begin{aligned}2\int_M (\operatorname{div} X)\, dV_g &= \int_{\widehat{M}} \widehat{\pi}^*\left((\operatorname{div} X)\, dV_g\right) = \int_{\widehat{M}} \left(\operatorname{div} \widehat{X}\right) |dV_{\widehat{g}}| \\ &= \int_{\widehat{M}} \left(\operatorname{div} \widehat{X}\right) dV_{\widehat{g}} = \int_{\partial \widehat{M}} \langle \widehat{X}, \widehat{N}\rangle_{\widehat{g}}\, dV_{\widetilde{g}} \\ &= \int_{\partial\widehat{M}} \langle \widehat{X}, \widehat{N}\rangle_{\widehat{g}}\, |dV_{\widetilde{g}}| = \int_{\partial\widehat{M}} \left(\widehat{\pi}|_{\partial\widehat{M}}\right)^* \left(\langle X, N\rangle_g\, dV_{\widetilde{g}}\right) \\ &= 2\int_{\partial M} \langle X, N\rangle_g\, dV_g.\end{aligned}$$

Dividing both sides by 2 yields (14.15). □

Problems

14-1. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ denote the 2-torus, defined by $w^2 + x^2 = y^2 + z^2 = 1$, with the orientation determined by its product structure (see Exercise 13.4). Compute $\int_{\mathbb{T}^2} \omega$, where ω is the following 2-form on $\mathbb{R}^4$:

$$\omega = xyz\, dw \wedge dy.$$

14-2. Let D denote the torus of revolution in $\mathbb{R}^3$ obtained by revolving the circle $(y-2)^2 + z^2 = 1$ around the z-axis (Example 11.23), with its induced Riemannian metric and with the orientation determined by the outward unit normal.

(a) Compute the surface area of D.
(b) Compute the integral over D of the function $f(x,y,z) = z^2 + 1$.
(c) Compute the integral over D of the 2-form $\omega = z\, dx \wedge dy$.

14-3. Let ω be the $(n-1)$-form on $\mathbb{R}^n \smallsetminus \{0\}$ defined by

$$\omega = |x|^{-n} \sum_{i=1}^{n} (-1)^{i-1} x^i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n. \qquad (14.16)$$

(a) Show that $\omega|_{\mathbb{S}^{n-1}}$ is the Riemannian volume form of $\mathbb{S}^{n-1}$ with respect to the round metric.
(b) Show that ω is closed but not exact on $\mathbb{R}^n \smallsetminus \{0\}$.

14-4. Define maps $F_+, F_- : \mathbb{B}^2 \to \mathbb{S}^2$ by

$$F_\pm(u,v) = \left(u, v, \pm\sqrt{1-u^2-v^2}\right).$$

If ω is a smooth 2-form on $\mathbb{S}^2$, show that

$$\int_{\mathbb{S}^2} \omega = \int_{\mathbb{B}^2} F_+^*\omega - \int_{\mathbb{B}^2} F_-^*\omega,$$

where the integrals on the right-hand side are defined as the limits as $R \nearrow 1$ of the integrals over $\overline{B}_R(0)$. Be sure to justify the limits.

14-5. Suppose $\widetilde{M}$ and M are smooth n-manifolds and $\pi : \widetilde{M} \to M$ is a smooth k-sheeted covering map.

(a) If $\widetilde{M}$ and M are oriented and π is orientation-preserving, show that $\int_{\widetilde{M}} \pi^*\omega = k\int_M \omega$ for any compactly supported n-form ω on M.
(b) If μ is any compactly supported density on M, show that $\int_{\widetilde{M}} \pi^*\mu = k\int_M \mu$.

14-6. If M is a compact, smooth, oriented manifold with boundary, show that there does not exist a smooth retraction of M onto its boundary. [Hint: Consider an orientation form on ∂M.]

14-7. Let (M,g) be a compact, oriented Riemannian manifold with boundary, let $\widetilde{g}$ denote the induced Riemannian metric on ∂M, and let N be the outward unit normal vector field along ∂M.

(a) Show that the divergence operator satisfies the following product rule for $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$:

$$\operatorname{div}(fX) = f \operatorname{div} X + \langle \operatorname{grad} f, X\rangle_g.$$

(b) Prove the following "integration by parts" formula:

$$\int_M \langle \operatorname{grad} f, X\rangle_g\, dV_g = \int_{\partial M} f\langle X, N\rangle_g\, dV_{\widetilde{g}} - \int_M (f \operatorname{div} X)\, dV_g.$$

(c) Explain what this has to do with integration by parts.

14-8. Let (M,g) be an oriented Riemannian manifold with or without boundary. The linear operator $\Delta : C^\infty(M) \to C^\infty(M)$ defined by $\Delta u = -\operatorname{div}(\operatorname{grad} u)$ is called the *Laplace operator* or *Laplacian*. A function $u \in C^\infty(M)$ is said to be *harmonic* if $\Delta u = 0$.

(a) If M is compact, prove *Green's identities*:

$$\int_M u\Delta v\, dV_g = \int_M \langle \operatorname{grad} u, \operatorname{grad} v\rangle_g\, dV_g - \int_{\partial M} u\, Nv\, dV_{\widetilde{g}},$$

$$\int_M (u\Delta v - v\Delta u)\, dV_g = \int_{\partial M} (v\, Nu - u\, Nv) dV_{\widetilde{g}},$$

where N and $\widetilde{g}$ are as in Problem 14-7.
(b) If M is compact and connected and $\partial M = \varnothing$, show that the only harmonic functions on M are the constants.
(c) If M is compact and connected, $\partial M \neq \varnothing$, and u, v are harmonic functions on M whose restrictions to ∂M agree, show that $u \equiv v$.

[Remark: There is no general agreement about the sign convention for the Laplacian on a Riemannian manifold, and many authors define the Laplacian to be the negative of the one we have defined here

and, when $\dim M = 3$,

$$\operatorname{curl} X = \left(*\,dX^\flat\right)^\#.$$

14-16. Let (M,g) be a compact, oriented Riemannian n-manifold. For $1 \le k \le n$, define a map $d^*\colon \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$ by $d^*\omega = (-1)^{n(k+1)+1} * d * \omega$, where $*$ is the Hodge star operator defined in Problem 14-12. Extend this definition to 0-forms by defining $d^*\omega = 0$ for $\omega \in \mathcal{A}^0(M)$.

(a) Show that $d^* \circ d^* = 0$.
(b) Show that the formula

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle_g \, dV_g$$

defines an inner product on $\mathcal{A}^k(M)$ for each k, where $\langle \cdot, \cdot \rangle_g$ is the pointwise inner product on forms defined in Problem 14-12.
(c) Show that $(d^*\omega, \eta) = (\omega, d\eta)$ for all $\omega \in \mathcal{A}^k(M)$ and $\eta \in \mathcal{A}^{k-1}(M)$.

14-17. On $\mathbb{R}^3$ with the Euclidean metric, show that the curl operator we have defined is given by the classical formula:

$$\operatorname{curl}\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}\right) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial}{\partial x} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\frac{\partial}{\partial z}.$$

14-18. Show that any finite product $M_1 \times \dots \times M_k$ of smooth manifolds with corners is again a smooth manifold with corners. Give a counterexample to show that a finite product of smooth manifolds with boundary need not be a smooth manifold with boundary.

14-19. Suppose M is a smooth manifold with corners, and let $\mathcal{C}$ denote the set of corner points of M. Show that $M \smallsetminus \mathcal{C}$ is a smooth manifold with boundary.

14-20. Show that the divergence operator on an oriented Riemannian manifold does not depend on the choice of orientation, and conclude that it is invariantly defined on all Riemannian manifolds.

14-21. Let M and N be compact, connected, oriented, smooth manifolds, and suppose $F, G\colon M \to N$ are diffeomorphisms. If F and G are homotopic, show that they are either both orientation-preserving or both orientation-reversing. [Hint: Use the Whitney approximation theorem and Stokes's theorem on $M \times I$.]

14-22. The Hairy Ball Theorem: *There exists a nowhere-vanishing vector field on $\mathbb{S}^n$ if and only if n is odd.* ("You cannot comb the hair on a ball.") Prove this by showing that the following are equivalent:

(a) There exists a nowhere-vanishing vector field on $\mathbb{S}^n$.
(b) There exists a continuous map $V\colon \mathbb{S}^n \to \mathbb{S}^n$ satisfying $V(x) \perp x$ (with respect to the Euclidean dot product on $\mathbb{R}^{n+1}$) for all $x \in \mathbb{S}^n$.
(c) The antipodal map $\alpha\colon \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to $\mathrm{Id}_{\mathbb{S}^n}$.
(d) The antipodal map $\alpha\colon \mathbb{S}^n \to \mathbb{S}^n$ is orientation-preserving.
(e) n is odd.

[Hint: Use Problems 8-7, 13-5, and 14-21.]