Assignment 5

Q1) Transversal $\Rightarrow$ clean intersection

Fix $p \in K \cap L \subset M$

clearly $T_p(K \cap L) \leq (T_p K) \cap (T_p L)$ as vector spaces

To prove K and L have clean intersection, it suffices to prove.

$$\dim T_p(K \cap L) = \dim (T_p K) \cap (T_p L).$$

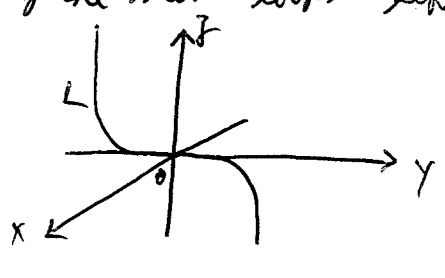
Let $k = \dim K$, $l = \dim L$, $m = \dim M$.

$$K \pitchfork L \Rightarrow T_pK + T_pL = T_pM \Rightarrow \dim (T_pK) \cap (T_pL) = k + l - m.$$

On the other hand we know that $\dim K \cap L = k + l - m$, hence $\dim T_p(K \cap L)$ $= k + l - m = \dim T_pK \cap T_pL$.

Clean intersection $\not\Rightarrow$ transversal

$M = \mathbb{R}^3$ with usual coord. x, y, z, $K =$ xy-plane, L embedding 1-dim submfd in yz-plane that looks like



Consider $p = 0 \in K \cap L$,

$$T_p K \cong \mathbb{R}^2, \quad T_p L \cong \mathbb{R}, \qquad T_p(K \cap L) \cong T_p L \cong \mathbb{R}$$

so K & L have clean intersection,

but clearly they do not intersect transversally at $p = 0$. //

Q2) (a) By def. of a smooth v.f., $X : M \to TM$ is 1-1.

Since X is a smooth global section of the natural projection

$$\begin{array}{cc} TM & (p, v_p) \\ \pi \downarrow & \downarrow \\ M & p \end{array}$$

, we see that X as a mapping is homeo. onto its image in TM.

It remains to show that $\forall p \in M$, $(dX)_p : T_p M \to T_{(p, X_p)}(TM)$ is 1-1.

Suppose $v \in T_p M$ is in kernel of $(dX)_p$, take a smooth curve $\gamma : \underset{0}{\overset{\cup}{I}} \to M$ s.t. $\gamma(0) = p$, $\gamma'(0) = v$, so $(X \circ \gamma)'(0) \in T_{(p, X_p)}(TM)$

is a zero tangent vector, ie $\forall f \in C^\infty(TM)$ we have

$$\left.\frac{d}{dt}\right|_{t=0} f(X \circ \gamma(t)) = 0$$

Now $\forall g \in C^\infty(M)$, $g \circ \pi \in C^\infty(TM)$, so

$$\left.\frac{d}{dt}\right|_{t=0} g(\gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} (g \circ \pi)(X \circ \gamma(t)) = 0$$

$$\Rightarrow \quad \underset{\underset{\displaystyle v.}{\|}}{\gamma'(0)} = 0 \in T_pM$$

$\Rightarrow$ dX_p is 1-1

$\Rightarrow$ X is an immersion

Hence X is an embedding.

Suppose X & Y are 2 such embeddings, then we can define the homotopy by $H: [0,1] \times M \to TM$

$$(t, p) \longmapsto (p, tX_p + (1-t)Y_p)$$

where X_p and Y_p denotes the tangent vectors of X & Y at p. //

(b) Compute $I_2(S^1, S^1)$:

Note that $\forall (x,y) \in S^1 \subset \mathbb{R}^2$, $V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ is a vector field on $\mathbb{R}^2$ tangent to S^1, and this vector field V never vanishes.

Let $Z \subset TS^1$ be the zero section, then $\#\{\text{Image}(V) \cap Z\} = 0$

$\Rightarrow I_2(S^1, S^1) = 0$

Compute $I_2(S^2, S^2)$: let $N = (0,0,1)$ north pole, $S = (0,0,-1)$ south pole.

$\phi_N : U_N \to \mathbb{R}^2$ stereo proj. from N, $\phi_S : U_S \to \mathbb{R}^2$ stereo proj. from S.

$\phi_N = (x,y)$ $\qquad\qquad$ $\phi_S = (v,w)$

define the v.f. $A = \begin{cases} -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} & \text{on } U_N \\ -w\frac{\partial}{\partial v} + v\frac{\partial}{\partial w} & \text{on } U_S \end{cases}$

so A only vanishes at N and S. If we can check

Image $A \pitchfork Z$ where $Z \subset TS^2$ is the zero section, then

$I_2(S^2, S^2) \equiv 2 \pmod 2$

$\equiv 0 \pmod 2$

For example at S, on U_N the mapping $A|_{U_N} : U_N \to TS^2$

can be realized as $A|_{U_N}(p) = \left(p, -y\frac{\partial}{\partial x}\Big|_p + x\frac{\partial}{\partial y}\Big|_p\right)$

so $(dA)_S\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x} + \frac{\partial}{\partial y_1}$, $(dA)_S\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y} - \frac{\partial}{\partial x_1}$.

here (x,y) coord on U_N . (x,y,x_1,y_1) coord on $TU_N \cong U_N \times \mathbb{R}^2$

Since $T_{(S,0)}(TS^2) \supseteq T_S S^2 = \text{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$

and $T_{(S,0)}(\text{Image } A) = \text{span}\left\{\frac{\partial}{\partial x} + \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y} - \frac{\partial}{\partial x_1}\right\}$,

and $T_{(S,0)}(TS^2) = \text{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\right\}$

we see that A is transversal to Z at S,

a similar computation show A is transversal to Z at N,

hence $I_2(S^2, S^2) = 0$

//.

Q3, (a)

$$\eta_{21}(e^{i\theta}) = \tau_{21} \oplus \tau_{21}'(e^{i\theta}) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } 0<\theta<\pi \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } -\pi<\theta<0 \end{cases}$$

(b) Recall $\mathcal{U}_1 = \{e^{i\theta} \mid -\pi < \theta < \pi\}$, $\mathcal{U}_2 = \{e^{i\theta} \mid 0 < \theta < 2\pi\}$

$\mathcal{U}_1 \cap \mathcal{U}_2 = A \cup B$ where $A = \{e^{i\theta} \mid 0<\theta<\pi\}$, $B = \{e^{i\theta} \mid \pi < \theta < 2\pi\}$

we just need to contruct 2 nonvanishing sections $F, G : S^1 \to E \oplus E$ that are everywhere linearly independent.

Define $F : S^1 \to E \oplus E$ by $F(e^{i\theta}) = \begin{cases} \sigma_1(e^{i\theta}) = [(e^{i\theta}, (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}))] & \text{if } e^{i\theta} \in \mathcal{U}_1 \\ \sigma_2(e^{i\theta}) = [(e^{i\theta}, (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}))] & \text{if } e^{i\theta} \in \mathcal{U}_2 \end{cases}$

check F is well-defined:

If $e^{i\theta} \in \mathcal{U}_1 \cap \mathcal{U}_2$, say when $e^{i\theta} \in A$, $(0<\theta<\pi)$ $\sigma_1(e^{i\theta}) = \sigma_2(e^{i\theta})$

when $e^{i\theta} \in B$, $(\pi < \theta < 2\pi)$ $\sigma_1(e^{i\theta}) = \sigma_1(e^{i(\theta - 2\pi)}) = [(e^{i\theta}, (\cos\frac{\theta-2\pi}{2}, \sin(\frac{\theta-2\pi}{2})))]$ $(-\pi < \theta - 2\pi < 0)$

$$= [(e^{i\theta}, \eta_{21}(e^{i\theta})(\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2}))]$$

$$= [(e^{i\theta}, (\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2}))] = \sigma_2(e^{i\theta})$$

$\Rightarrow F$ is well-defined

Next define $G(e^{i\theta}) = \begin{cases} \alpha_1(e^{i\theta}) = [(e^{i\theta}, (-\sin\frac{\theta}{2}, \cos\frac{\theta}{2}))] & \text{if } e^{i\theta} \in U_1 \\ \alpha_2(e^{i\theta}) = [(e^{i\theta}, (-\sin\frac{\theta}{2}, \cos\frac{\theta}{2}))] & \text{if } e^{i\theta} \in U_2 \end{cases}$

again check G is well-defined as above.

Note F & G are linearly independent everywhere, so we are done //

5-7) $\sigma_N : U_N \longrightarrow \mathbb{R}^2$ $\qquad$ $\sigma_S : U_S \longrightarrow \mathbb{R}^2$

$(x_1, x_2, x_3) \longmapsto \dfrac{(x_1, x_2)}{1-x_3}$ $\qquad$ $(x_1, x_2, x_3) \longmapsto \dfrac{(x_1, x_2)}{1+x_3}$

$\sigma_N \circ \sigma_S^{-1}(u_1, u_2) = \left(\dfrac{u_1}{u_1^2+u_2^2}, \dfrac{u_2}{u_1^2+u_2^2} \right)$ let $u = (u_1, u_2)$

so $$D_{(u_1,u_2)} \sigma_N \circ \sigma_S^{-1} = \begin{pmatrix} \dfrac{u_2^2 - u_1^2}{|u|^4} & \dfrac{-2u_1u_2}{|u|^4} \\ \dfrac{-2u_1u_2}{|u|^4} & \dfrac{u_1^2 - u_2^2}{|u|^4} \end{pmatrix}$$

//

6-4) $F^* df = F^*(2x\,dx + 2y\,dy + 2z\,dz)$

$$= 2\left(\frac{2u}{u^2+v^2+1}\, d\left(\frac{2u}{u^2+v^2+1}\right) + \frac{2v}{u^2+v^2+1}\, d\left(\frac{2v}{u^2+v^2+1}\right) + \frac{u^2+v^2-1}{u^2+v^2+1}\, d\left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)\right)$$

$$= 0.$$

$$d(f \circ F) = d(1) = 0. \quad //$$

11-4) $\sigma_{i_1 \cdots i_k} = \sigma_{i_1 \cdots i_k}\, dx^{i_1} \otimes \cdots \otimes dx^{i_k}(E_{i_1}, \cdots, E_{i_k})$

where E_i's are basis dual to dx^i's

$$= \tilde{\sigma}_{j_1 \cdots j_k}\, d\tilde{x}^{j_1} \otimes \cdots \otimes d\tilde{x}^{j_k}(E_{i_1} \cdots E_{i_k})$$

$$= \tilde{\sigma}_{j_1 \cdots j_k} \frac{\partial \tilde{x}^{j_1}}{\partial x_{i_1}} \cdots \frac{\partial \tilde{x}^{j_k}}{\partial x_{i_k}}.$$

11-5) If $\sigma^{i_1' \cdots i_l'}_{i_1 \cdots i_k} \frac{\partial}{\partial x_{i_1'}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_l'}} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_k}$

$$= \tilde{\sigma}^{j_1' \cdots j_l'}_{j_1 \cdots j_k} \frac{\partial}{\partial \tilde{x}_{j_1'}} \otimes \cdots \otimes \frac{\partial}{\partial \tilde{x}_{j_l'}} \otimes d\tilde{x}^{j_1} \otimes \cdots \otimes d\tilde{x}^{j_k}$$

Apply both sides by $(dx^{i_1'}, \cdots, dx^{i_l'}, \frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_k}})$

We then have

$$\sigma^{i_1' \cdots i_l'}_{i_1 \cdots i_k} = \tilde{\sigma}^{j_1' \cdots j_l'}_{j_1 \cdots j_k} \frac{\partial x^{i_1'}}{\partial \tilde{x}^{j_1'}} \cdots \frac{\partial x^{i_l'}}{\partial \tilde{x}^{j_l'}} \frac{\partial \tilde{x}^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial \tilde{x}^{j_k}}{\partial x^{i_k}}$$