By the result of Problem 5-4, there is a smooth real line bundle $F \to \mathbb{S}^1$ that is trivial over U and V, and has τ as transition function. Show that F is smoothly isomorphic to the Möbius bundle of Example 5.2.

5-7. Compute the transition function for $T\mathbb{S}^2$ associated with the two local trivializations determined by stereographic coordinates.

5-8. Let $\pi\colon E \to M$ be a smooth vector bundle of rank k, and suppose $\sigma_1, \dots, \sigma_m$ are independent smooth local sections over an open subset $U \subset M$. Show that for each $p \in U$ there are smooth sections $\sigma_{m+1}, \dots, \sigma_k$ defined on some neighborhood V of p such that $(\sigma_1, \dots, \sigma_k)$ is a smooth local frame for E over $U \cap V$.

5-9. Suppose E and E' are vector bundles over a smooth manifold M, and $F\colon E \to E'$ is a bijective bundle map over M. Show that F is a bundle isomorphism.

5-10. Consider the following vector fields on $\mathbb{R}^4$:

$$X_1 = -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2} + x^4\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^4},$$
$$X_2 = -x^3\frac{\partial}{\partial x^1} - x^4\frac{\partial}{\partial x^2} + x^1\frac{\partial}{\partial x^3} + x^2\frac{\partial}{\partial x^4},$$
$$X_3 = -x^4\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^3} + x^1\frac{\partial}{\partial x^4}.$$

Show that there are smooth vector fields V_1, V_2, V_3 on $\mathbb{S}^3$ such that V_j is ι-related to X_j for $j = 1, 2, 3$, where $\iota\colon \mathbb{S}^3 \hookrightarrow \mathbb{R}^4$ is inclusion. Conclude that $\mathbb{S}^3$ is parallelizable.

5-11. Let V be a finite-dimensional vector space, and let $G_k(V)$ be the Grassmannian of k-dimensional subspaces of V. Let T be the disjoint union of all these k-dimensional subspaces:

$$T = \coprod_{S \in G_k(V)} S;$$

and let $\pi\colon T \to G_k(V)$ be the natural map sending each point $x \in S$ to S. Show that T has a unique smooth manifold structure making it into a smooth rank-k vector bundle over $G_k(V)$, with π as projection and with the vector space structure on each fiber inherited from V. [Remark: T is sometimes called the *tautological vector bundle* over $G_k(V)$, because the fiber over each point $S \in G_k(V)$ is S itself.]

5-12. Show that the tautological vector bundle over $G_1(\mathbb{R}^2)$ is isomorphic to the Möbius bundle. (See Problems 5-2, 5-6, and 5-11.)

5-13. Let V_0 be the category whose objects are finite-dimensional real vector spaces and whose morphisms are linear isomorphisms. If $\mathcal{F}$ is a covariant functor from V_0 to itself, for each finite-dimensional vector space V we get a map $\mathcal{F}\colon \mathrm{GL}(V) \to \mathrm{GL}(\mathcal{F}(V))$ sending each isomorphism $A\colon V \to V$ to the induced isomorphism $\mathcal{F}(A)\colon \mathcal{F}(V) \to \mathcal{F}(V)$. We say $\mathcal{F}$ is a *smooth functor* if this map is smooth for every V. If $\mathcal{F}\colon \mathsf{V}_0 \to \mathsf{V}_0$ is a smooth functor, show that for every smooth vector bundle $E \to M$ there is a smooth vector bundle $\mathcal{F}(E) \to M$ whose fiber at each point $p \in M$ is $\mathcal{F}(E_p)$.

Example 6.32. Let ω be a smooth covector field on $\mathbb{R}^3$, say

$$\omega = e^{y^2}\,dx + 2xye^{y^2}\,dy - 2z\,dz.$$

You can check that ω is closed. For f to be a potential for ω, we must have

$$\frac{\partial f}{\partial x} = e^{y^2}, \quad \frac{\partial f}{\partial y} = 2xye^{y^2}, \quad \frac{\partial f}{\partial z} = -2z.$$

Holding y and z fixed and integrating the first equation with respect to x, we obtain

$$f(x,y,z) = \int e^{y^2}\,dx = xe^{y^2} + C_1(y,z),$$

where the "constant" of integration $C_1(y,z)$ may depend on the choice of (y,z). Now the second equation implies

$$2xye^{y^2} = \frac{\partial}{\partial y}\left(xe^{y^2} + C_1(y,z)\right) = 2xye^{y^2} + \frac{\partial C_1}{\partial y},$$

which forces $\partial C_1/\partial y = 0$, so C_1 is actually a function of z only. Finally, the third equation implies

$$-2z = \frac{\partial}{\partial z}\left(xe^{y^2} + C_1(z)\right) = \frac{\partial C_1}{\partial z},$$

from which we conclude that $C_1(z) = -z^2 + C$, where C is an arbitrary constant. Thus a potential function for ω is given by $f(x,y,z) = xe^{y^2} - z^2$. Any other potential differs from this one by a constant.

You should convince yourself that the formal procedure we followed in this example is equivalent to choosing an arbitrary base point $c \in \mathbb{R}^3$, and defining $f(x,y,z)$ by integrating ω along a path from c to (x,y,z) consisting of three straight line segments parallel to the axes. This works for any closed covector field defined on an open rectangle in $\mathbb{R}^n$ (which we know must be exact, because a rectangle is convex). In practice, once a formula is found for f on some open rectangle, the same formula typically works for the entire domain. (This is because most of the covector fields for which one can explicitly compute the integrals as we did above are real-analytic, and real-analytic functions are determined by their behavior in any open set.)

Problems

6-1. (a) If V and W are finite-dimensional vector spaces and $A\colon V \to W$ is any linear map, show that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\;A\;} & W \\ {\scriptstyle \xi_V}\downarrow & & \downarrow{\scriptstyle \xi_W} \\ V^{**} & \xrightarrow[(A^*)^*]{} & W^{**}, \end{array}$$

where ξ_V and ξ_W denote the isomorphisms defined by (6.3) for V and W, respectively.

(b) Show that there does not exist a rule that assigns to each finite-dimensional vector space V an isomorphism $\beta_V\colon V \to V^*$ in such a way that for every linear map $A\colon V \to W$, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\;A\;} & W \\ {\scriptstyle \beta_V}\downarrow & & \downarrow{\scriptstyle \beta_W} \\ V^{*} & \xleftarrow[A^*]{} & W^{*}. \end{array}$$

6-2. (a) If $F\colon M \to N$ is a smooth map, show that $F^*\colon T^*N \to T^*M$ is a smooth bundle map.

(b) Show that the assignment $M \mapsto T^*M$, $F \mapsto F^*$ defines a contravariant functor from the category of smooth manifolds to the category of smooth vector bundles.

6-3. If M is a smooth manifold, show that T^*M is a trivial bundle if and only if TM is trivial.

6-4. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x,y,z) = x^2 + y^2 + z^2$, and let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the following map (the inverse of stereographic projection):

$$F(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

Compute F^*df and $d(f \circ F)$ separately, and verify that they are equal.

6-5. In each of the cases below, M is a smooth manifold and $f\colon M \to \mathbb{R}$ is a smooth function. Compute the coordinate representation for df, and determine the set of all points $p \in M$ at which $df_p = 0$.

(a) $M = \{(x,y) \in \mathbb{R}^2 : x > 0\}$; $f(x,y) = x/(x^2+y^2)$. Use standard coordinates (x,y).

(b) M and f are as in part (a); this time use polar coordinates (r,θ).

(c) $M = S^2 \subset \mathbb{R}^3$; $f(p) = z(p)$ (the z-coordinate of p, thought of as a point in $\mathbb{R}^3$). Use stereographic coordinates.

commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{A} & Y \\ {\scriptstyle \widetilde{\pi}}\downarrow & \nearrow {\scriptstyle \widetilde{A}} & \\ Z. & & \end{array}$$

Then there is a unique isomorphism $\Phi\colon V \otimes W \to Z$ such that $\widetilde{\pi} = \Phi \circ \pi$, where $\pi\colon V \times W \to V \otimes W$ is the canonical projection. [Remark: This shows that the details of the construction used to define the tensor product are irrelevant, as long as the resulting space satisfies the characteristic property.]

11-2. If V is any finite-dimensional real vector space, prove that there are canonical isomorphisms $\mathbb{R} \otimes V \cong V \cong V \otimes \mathbb{R}$.

11-3. Let V and W be finite-dimensional real vector spaces. Prove that there is a canonical (basis-independent) isomorphism between $V^* \otimes W$ and the space $\operatorname{Hom}(V, W)$ of linear maps from V to W.

11-4. Let M be a smooth n-manifold, and let σ be a smooth covariant k-tensor field on M. If $(U, (x^i))$ and $(\widetilde{U}, (\widetilde{x}^j))$ are overlapping smooth charts on M, we can write

$$\sigma = \sigma_{i_1 \dots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} = \widetilde{\sigma}_{j_1 \dots j_k} d\widetilde{x}^{j_1} \otimes \cdots \otimes d\widetilde{x}^{j_k}.$$

Compute a transformation law analogous to (6.7) expressing the component functions $\sigma_{i_1 \dots i_k}$ in terms of $\widetilde{\sigma}_{j_1 \dots j_k}$.

11-5. Generalize the coordinate transformation law of Problem 11-4 to mixed tensors of any rank.

11-6. Suppose $F\colon M \to N$ is a diffeomorphism. For any pair of nonnegative integers k, l, show that there are smooth bundle isomorphisms $F_*\colon T^k_l M \to T^k_l N$ and $F^*\colon T^k_l N \to T^k_l M$ satisfying

$$\begin{aligned} F_* S\left(X_1, \dots, X_k, \omega^1, \dots, \omega^l\right) &= S\left(F_*^{-1} X_1, \dots, F_*^{-1} X_k, F^* \omega^1, \dots, F^* \omega^l\right), \\ F^* S\left(X_1, \dots, X_k, \omega^1, \dots, \omega^l\right) &= S\left(F_* X_1, \dots, F_* X_k, F^{-1*} \omega^1, \dots, F^{-1*} \omega^l\right). \end{aligned}$$

11-7. Let M be a smooth manifold.

(a) Given a smooth covariant k-tensor field $\tau \in \mathcal{T}^k(M)$, show that the map $\mathcal{T}(M) \times \cdots \times \mathcal{T}(M) \to C^\infty(M)$ defined by

$$(X_1, \dots, X_k) \mapsto \tau(X_1, \dots, X_k)$$

is multilinear over $C^\infty(M)$, in the sense that for any smooth functions $f, f' \in C^\infty(M)$ and smooth vector fields X_i, X_i',

$$\begin{aligned} \tau(X_1, \dots, f X_i + f' X_i', \dots, X_k) \\ = f \tau(X_1, \dots, X_i, \dots, X_k) + f' \tau(X_1, \dots, X_i', \dots, X_k). \end{aligned}$$

(b) Show that a map

$$\widetilde{\tau}\colon \mathcal{T}(M) \times \cdots \times \mathcal{T}(M) \to C^\infty(M)$$

is induced by a smooth tensor field as above if and only if it is multilinear over $C^\infty(M)$.

11-8. Let V be an n-dimensional real vector space. Show that

$$\dim \Sigma^k(V) = \binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}.$$

11-9. (a) Let T be a covariant k-tensor on a finite-dimensional real vector space V. Show that $\operatorname{Sym} T$ is the unique symmetric k-tensor satisfying

$$(\operatorname{Sym} T)(X, \dots, X) = T(X, \dots, X)$$

for all $X \in V$.

(b) Show that the symmetric product is associative: For all symmetric tensors R, S, T,

$$(RS)T = R(ST).$$

[Hint: Use part (a).]

(c) If $\omega^1, \dots, \omega^k$ are covectors, show that

$$\omega^1 \cdots \omega^k = \frac{1}{k!} \sum_{\sigma \in S_k} \omega^{\sigma(1)} \otimes \cdots \otimes \omega^{\sigma(k)}.$$

11-10. Let $\mathring{g} = \overline{g}|_{\mathbb{S}^n}$ denote the round metric on the n-sphere, i.e., the metric induced from the Euclidean metric by the usual inclusion $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$.

(a) Derive an expression for $\mathring{g}$ in stereographic coordinates by computing the pullback $(\sigma^{-1})^* \overline{g}$.

(b) In the case $n = 2$, do the analogous computation in spherical coordinates $(x, y, z) = (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi)$.

11-11. Let M be any smooth manifold.

(a) Show that TM and T^*M are isomorphic vector bundles.

(b) Show that the isomorphism of part (a) is *not* canonical, in the following sense: There does not exist a rule that assigns to every smooth manifold M a bundle isomorphism $\lambda_M\colon TM \to T^*M$ in such a way that for every diff...