Math 518 Differentiable Manifolds I

Assignment 5, Due Thursday November 18

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1. Let K and L be embedded submanifolds of a manifold M and suppose that their intersection $K \cap L$ is also an embedded submanifold of M. Then K and L are said to have **clean intersection** if for each $p \in K \cap L$ we have
$$T_p(K \cap L) = T_pK \cap T_pL.$$
Can the intersection of embedded submanifolds be transverse but not clean? Can it be clean but not transverse? Give examples or proofs as necessary.

2. (a) Prove that any vector field $X\colon M \to TM$ is an embedding of M and that all such embeddings are smoothly homotopic.

   (b) Recall that given smooth maps $F\colon K^k \to M^m$ and $G\colon L^l \to M^m$ where $k + l = m$ we defined the mod 2 intersection number of F and G to be
$$I_2(F, G) = \#(K_{F'} \times_G L) \mod 2$$
where F' is homotopic to F and transverse to G. When F and G are embeddings let's denote $I_2(F, G)$ instead by $I_2(K, L)$. If $2k = m$ we can then define the mod 2 self-intersection number of K to be $I_2(K, K)$.

   Compute the mod 2 self-intersection number of the zero section $K \to TK$ for the manifolds $K = S^1, S^2$. Hint: use part (a) to get an embedding of K transverse to the zero section.

   (c) BONUS: Compute the mod 2 self-intersection number of the zero section $\mathbb{R}P^2 \to T(\mathbb{R}P^2)$. Deduce that every smooth vector field on $\mathbb{R}P^2$ must have a zero.

3. Let E be the Möbius band described as the total space of a one dimension vector bundle over S^1, as in class.

   (a) Using our transition maps for E, write down transition maps for the Whitney sum $E \oplus E$.

   (b) Show that the Whitney sum $E \oplus E$ is a trivial two dimension vector bundle over S^1. Hint: It suffices to construct two nonvanishing sections of $E \oplus E$ which are everywhere linearly independent.

Problems from the text: 5-7, 6-4, 11-4, 11-5.