disadvantage of the germ approach is simply that it adds an additional level of complication to an already highly abstract definition.

Another common approach to defining T_pM is to define an intrinsic equivalence relation on the set of smooth curves in M starting at p, which amounts to "having the same tangent vector," and to define a tangent vector as an equivalence class of curves. For example, one such equivalence relation is the following: If $\gamma_1 \colon J_1 \to M$ and $\gamma_2 \colon J_2 \to M$ are two smooth curves such that $\gamma_1(0) = \gamma_2(0) = p$, then we say $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Problem 3-8 shows that the set of equivalence classes is in one-to-one correspondence with T_pM. This definition has the advantage of being geometrically more intuitive, but it has the serious drawback that the existence of a vector space structure on T_pM is not at all obvious.

Yet another approach to defining the tangent space is based on the transformation rule (3.9) for the components of tangent vectors in coordinates. One defines a tangent vector at a point $p \in M$ to be a rule that assigns a vector $(X^1, \dots, X^n) \in \mathbb{R}^n$ to each smooth coordinate chart containing p, with the property that the vectors assigned to overlapping charts transform according to (3.9). (This is, in fact, the oldest definition of all, and many physicists are still apt to define tangent vectors this way.)

It is a matter of individual taste which of the various characterizations of T_pM one chooses to take as the definition. The modern definition we have chosen, however abstract it may seem at first, has several advantages: It is relatively concrete (tangent vectors are actual derivations of $C^\infty(M)$, with no equivalence classes involved); it makes the vector space structure on T_pM obvious; and it leads to straightforward coordinate-independent definitions of many of the other geometric objects we will be studying.

Problems

3-1. Suppose M and N are smooth manifolds with M connected, and $F \colon M \to N$ is a smooth map such that $F_* \colon T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$. Show that F is a constant map.

3-2. Let $M_1, \dots, M_k$ be smooth manifolds, and let $\pi_j \colon M_1 \times \dots \times M_k \to M_j$ be the projection onto the jth factor. Show that for any choices of points $p_i \in M_i$, $i = 1, \dots, k$, the map

$$\alpha \colon T_{(p_1,\dots,p_k)}(M_1 \times \dots \times M_k) \to T_{p_1}M_1 \oplus \dots \oplus T_{p_k}M_k$$

defined by

$$\alpha(X) = (\pi_{1*}X, \dots, \pi_{k*}X)$$

is an isomorphism. [Remark: Using this isomorphism, we can routinely identify T_pM and T_qN, for example, as subspaces of $T_{(p,q)}(M \times N)$.]

3-3. If a nonempty smooth n-manifold is diffeomorphic to an m-manifold, prove that $n = m$.

3-4. Let $C \subset \mathbb{R}^2$ be the unit circle, and let $S \subset \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin:

$$S = \{(x, y) : \max(|x|, |y|) = 1\}.$$

Show that there is a homeomorphism $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(C) = S$, but there is no *diffeomorphism* with the same property. [Hint: Consider what F does to the tangent vector to a suitable curve in C.]

3-5. Consider $\mathbb{S}^3$ as a subset of $\mathbb{C}^2$ under the usual identification of $\mathbb{C}^2$ with $\mathbb{R}^4$. For each $z = (z^1, z^2) \in \mathbb{S}^3$, define a curve $\gamma_z \colon \mathbb{R} \to \mathbb{S}^3$ by

$$\gamma_z(t) = \left(e^{it}z^1, e^{it}z^2\right).$$

(a) Compute the coordinate representation of $\gamma_z(t)$ in stereographic coordinates, and use this to show that γ_z is a smooth curve.
(b) Compute $\gamma_z'(t)$ in stereographic coordinates, and show that it is never zero.

3-6. Let G be a Lie group.

(a) Let $m \colon G \times G \to G$ denote the multiplication map. Identifying $T_{(e,e)}(G \times G)$ with $T_eG \oplus T_eG$ as in Problem 3-2, show that $m_* \colon T_eG \oplus T_eG \to T_eG$ is given by $m_*(X, Y) = X + Y$ [Hint: Compute $m_*(X, 0)$ and $m_*(0, Y)$ separately using (3.10).]
(b) Let $i \colon G \to G$ denote the inversion map. Show that $i_* \colon T_eG \to T_eG$ is given by $i_*X = -X$.

3-7. Let M be a smooth manifold. For any point $p \in M$, let C_p^∞ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_p$ denote the vector space of derivations of C_p^∞ at p. Show that T_pM is naturally isomorphic to $\mathcal{D}_p$.

3-8. Let M be a smooth manifold and $p \in M$. Let $\mathcal{C}_p$ denote the set of smooth curves $\gamma \colon J \to M$ such that $0 \in J$ and $\gamma(0) = p$. Define an equivalence relation on $\mathcal{C}_p$ by saying that $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p, and let $\mathcal{V}_p$ denote the set of equivalence classes. Show that the map $\Phi \colon \mathcal{V}_p \to T_pM$ defined by $\Phi[\gamma] = \gamma'(0)$ is well-defined and yields a one-to-one correspondence between $\mathcal{V}_p$ and T_pM.

This proves that $\pi(W)$ is open, so (a) holds. Because a surjective open map is automatically a quotient map, (c) follows from (a). □

The next three propositions provide important tools that we will use frequently when studying submersions. The general philosophy of the proofs is this: To "push" a smooth object (such as a smooth map) down via a submersion, pull it back via local sections.

Proposition 7.17. *Suppose M, N, and P are smooth manifolds, $\pi\colon M \to N$ is a surjective submersion, and $F\colon N \to P$ is any map. Then F is smooth if and only if $F \circ \pi$ is smooth:*

$$\begin{array}{ccc} M & & \\ {\scriptstyle\pi}\downarrow & \searrow^{F\circ\pi} & \\ N & \xrightarrow[F]{} & P. \end{array}$$

Proof. If F is smooth, then $F \circ \pi$ is smooth by composition. Conversely, suppose that $F \circ \pi$ is smooth, and let $q \in N$ be arbitrary. For any $p \in \pi^{-1}(q)$, Proposition 7.16(b) guarantees the existence of a neighborhood U of q and a smooth local section $\sigma\colon U \to M$ of π such that $\sigma(q) = p$. Then $\pi \circ \sigma = \mathrm{Id}_U$ implies

$$F|_U = F|_U \circ \mathrm{Id}_U = F|_U \circ (\pi \circ \sigma) = (F \circ \pi) \circ \sigma,$$

which is a composition of smooth maps. This shows that F is smooth in a neighborhood of each point, so it is smooth. □

The next proposition gives a very general sufficient condition under which a smooth map can be "pushed down" by a submersion.

Proposition 7.18 (Passing Smoothly to the Quotient). *Suppose $\pi\colon M \to N$ is a surjective submersion. If $F\colon M \to P$ is a smooth map that is constant on the fibers of π, then there is a unique smooth map $\widetilde{F}\colon N \to P$ such that $\widetilde{F} \circ \pi = F$:*

$$\begin{array}{ccc} M & & \\ {\scriptstyle\pi}\downarrow & \searrow^{F} & \\ N & \xrightarrow[\widetilde{F}]{} & P. \end{array}$$

Proof. Clearly, if $\widetilde{F}$ exists, it will have to satisfy $\widetilde{F}(q) = F(p)$ whenever $p \in \pi^{-1}(q)$. We use this to *define* $\widetilde{F}$: Given $q \in N$, let $\widetilde{F}(q) = F(p)$, where $p \in M$ is any point in the fiber over q. (Such a point exists because we are assuming that π is surjective.) This is well-defined because F is constant on the fibers of π, and it satisfies $\widetilde{F} \circ \pi = F$ by construction. Thus $\widetilde{F}$ is smooth by Proposition 7.17. □

Our third proposition can be interpreted as a uniqueness result for smooth manifolds defined as quotients of other smooth manifolds by submersions.

Proposition 7.19 (Uniqueness of Smooth Quotients). *Suppose $\pi_1\colon M \to N_1$ and $\pi_2\colon M \to N_2$ are surjective submersions that are constant on each other's fibers. Then there exists a unique diffeomorphism $F\colon N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$:*

$$\begin{array}{ccc} & M & \\ {\scriptstyle\pi_1}\swarrow & & \searrow{\scriptstyle\pi_2} \\ N_1 & \xrightarrow[F]{} & N_2. \end{array}$$

◇ **Exercise 7.5.** Prove Proposition 7.19.

Problems

7-1. Using the covering map $\pi\colon \mathbb{R}^2 \to \mathbb{T}^2$ defined by $\pi(\varphi, \theta) = (e^{i\varphi}, e^{i\theta})$, show that the immersion $X\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined in Example 7.1(f) descends to an embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$.

7-2. Define a map $F\colon \mathbb{S}^2 \to \mathbb{R}^4$ by

$$F(x, y, z) = \left(x^2 - y^2, xy, xz, yz\right).$$

Using the smooth covering map $p\colon \mathbb{S}^2 \to \mathbb{RP}^2$ described in Example 2.5(d) and Problem 2-9, show that F descends to a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

7-3. Let $\gamma\colon \mathbb{R} \to \mathbb{T}^2$ be the curve of Example 7.3. Show that the image set $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$.

7-4. Suppose M is a smooth manifold, $p \in M$, and $y^1, \dots, y^n$ are smooth real-valued functions defined on a neighborhood of p in M.

(a) If $dy^1|_p, \dots, dy^n|_p$ form a basis for T_p^*M, show that $(y^1, \dots, y^n)$ are smooth coordinates for M in some neighborhood of p.

(b) If $dy^1|_p, \dots, dy^n|_p$ are independent, show that there are real-valued functions $y^{n+1}, \dots, y^m$ such that $(y^1, \dots, y^m)$ are smooth coordinates for M in some neighborhood of p.

(c) If $dy^1|_p, \dots, dy^n|_p$ span T_p^*M, show that there are indices $i_1, \dots, i_k$ such that $(y^{i_1}, \dots, y^{i_k})$ are smooth coordinates for M in some neighborhood of p.

7-5. Let M be a smooth compact manifold. Show that there is no submersion $F\colon M \to \mathbb{R}^k$ for any $k > 0$.

7-6. Suppose $\pi\colon M \to N$ is a smooth map such that every point of M is in the image of a smooth local section of π. Show that π is a submersion.

(iii) With the vector space structure on each D_p inherited from E_p and the projection $\pi|_D : D \to M$, D is a smooth vector bundle over M.

Note that the condition that D be a vector bundle over M implies that the projection $\pi|_D : D \to M$ must be surjective, and that all the fibers D_p must have the same dimension.

◇ **Exercise 8.7.** If $D \subset E$ is a smooth subbundle, show that the inclusion map $\iota : D \hookrightarrow E$ is a smooth bundle map over M.

The following lemma gives a convenient condition for checking that a collection of subspaces $\{D_p \subset E_p : p \in M\}$ is a smooth subbundle.

Lemma 8.41 (Local Frame Criterion for Subbundles). *Let $\pi : E \to M$ be a smooth vector bundle, and suppose for each $p \in M$ we are given an m-dimensional linear subspace $D_p \subset E_p$. Then $D = \coprod_{p \in M} D_p \subset E$ is a smooth subbundle if and only if the following condition is satisfied:*

Each point $p \in M$ has a neighborhood U on which there are smooth local sections $\sigma_1, \dots, \sigma_m : U \to E$ such that $\sigma_1|_q, \dots, \sigma_m|_q$ form a basis for D_q at each $q \in U$. (8.4)

Proof. If D is a smooth subbundle, then by definition any $p \in M$ has a neighborhood U over which there exists a smooth local trivialization of D, and Example 5.9 shows that there exists a smooth local frame for D over any such set U. Such a local frame is by definition a collection of smooth sections $\tau_1, \dots, \tau_m : U \to D$ whose images form a basis for D_p at each point $p \in U$. The smooth sections of E that we seek are obtained simply by composing with the inclusion map $\iota : D \hookrightarrow E$: $\sigma_j = \iota \circ \tau_j$.

Conversely, suppose that D satisfies (8.4). Condition (ii) in the definition of a subbundle is true by hypothesis, so we need to show that D satisfies conditions (i) and (iii).

To prove that D is an embedded submanifold, it suffices to show that each point $p \in M$ has a neighborhood U such that $D \cap \pi^{-1}(U)$ is an embedded submanifold of $\pi^{-1}(U) \subset E$. Given $p \in M$, let $\sigma_1, \dots, \sigma_m$ be smooth local sections of E defined on a neighborhood of p and satisfying (8.4). By the result of Problem 5-8, we can extend these to a smooth local frame $(\sigma_1, \dots \sigma_k)$ for E over some neighborhood U of p. By Proposition 5.10, this local frame is associated with a smooth local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, defined by

$$\Phi\left(s^i \sigma_i|_q\right) = \left(q, \left(s^1, \dots, s^k\right)\right).$$

This map Φ takes $D \cap \pi^{-1}(U)$ to $U \times \mathbb{R}^m = \left\{\left(q, \left(s^1, \dots, s^m, 0, \dots, 0\right)\right)\right\} \subset U \times \mathbb{R}^k$, which is obviously an embedded submanifold. Moreover, the map $\Phi|_{D \cap \pi^{-1}(U)} : D \cap \pi^{-1}(U) \to U \times \mathbb{R}^m$ is a smooth local trivialization of D, showing that D is itself a smooth vector bundle. □

Example 8.42. Suppose M is any parallelizable manifold, and let $(E_1, \dots, E_n)$ be a smooth global frame for M. If $1 \le k \le n$, the subset $D \subset TM$ defined by $D_p = \operatorname{span}(E_1, \dots, E_k)$ is a smooth subbundle of TM.

Problems

8-1. Consider the map $F : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$F(x, y, s, t) = \left(x^2 + y, x^2 + y^2 + s^2 + t^2 + y\right).$$

Show that $(0, 1)$ is a regular value of F, and that the level set $F^{-1}(0, 1)$ is diffeomorphic to $\mathbb{S}^2$.

8-2. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) = x^3 + xy + y^3.$$

Which level sets of F are embedded submanifolds of $\mathbb{R}^2$?

8-3. Show that the image of the curve $\gamma : (-\pi/2, 3\pi/2) \to \mathbb{R}^2$ of Example 7.2 is not an embedded submanifold of $\mathbb{R}^2$. [Be careful: This is not the same as showing that γ is not an embedding.]

8-4. Let $S \subset \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin (see Problem 3-4). Show that S does not have a topology and smooth structure in which it is an immersed submanifold of $\mathbb{R}^2$.

8-5. Let $\gamma : \mathbb{R} \to \mathbb{T}^2$ be the curve of Example 7.3. Show that $\gamma(\mathbb{R})$ is not an embedded submanifold of the torus.

8-6. Our definition of Lie groups included the requirement that both the multiplication map and the inversion map are smooth. Show that smoothness of the inversion map is redundant: If G is a smooth manifold with a group structure such that the multiplication map $m : G \times G \to G$ is smooth, then G is a Lie group.

8-7. This problem generalizes the result of Problem 4-10 to higher dimensions. For any integer $n \ge 1$, define a vector field on $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ by $V_z = \gamma_z'(0)$, where $\gamma_z : \mathbb{R} \to \mathbb{S}^{2n-1}$ is the curve $\gamma_z(t) = e^{it} z$. Show that V is smooth and nowhere vanishing.

8-8. Show that an embedded submanifold is closed if and only if the inclusion map is proper.

8-9. For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by

$$M_a = \left\{(x, y) : y^2 = x(x-1)(x-a)\right\}.$$