Assignment 3

Q1) For example if $\varphi_N^{-1} : \mathbb{R}^3 \longrightarrow S^3 \setminus \{N\}$ where $N = (0,0,0,1)$

st. $\varphi_N^{-1}(u_1, u_2, u_3) = \left(\frac{2u_1}{|u|^2+1}, \frac{2u_2}{|u|^2+1}, \frac{2u_3}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1} \right)$

If $\phi_1 : U_1 \subseteq \mathbb{C}P^1 \longrightarrow \mathbb{C}$ st. $\phi_1[z_1 : z_2] = \frac{z_2}{z_1}$ where

$U_1 = \{ [z_1 : z_2] \mid z_1 \neq 0 \}$

Then $\phi_1 \circ p \circ \varphi_N^{-1}(u_1, u_2, u_3) = \dfrac{2u_3 + i(|u|^2 - 1)}{2u_1 + 2iu_2}$

which is smooth since $u_1 + iu_2 \neq 0$ in the domain of def.

Q2) Construct $V_1 \in \mathfrak{X}(S^2)$ st. it only vanishes on N and S:

Let (ϕ_N, U_N), (ϕ_S, U_S) be the stereographic coord. chart on S^2 (ϕ_N stereo. proj. from N and ϕ_S stereo proj. from S)

Denote $\phi_N = (x, y)$, $\phi_S = (u, v)$

Take $V_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on U_N

$V_1 = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$ on U_S

so V_1 only vanishes at N and S, by calculating the formula for the change of coordinates on $U_N \cap U_S$, V_1 is well-defined smooth v.f. on S^2.

Construct $V_2 \in \mathcal{X}(S^2)$ st. it only vanishes at N:

Take $V_2 = \frac{\partial}{\partial x}$ on U_N

$V_2 = (v^2 - u^2)\frac{\partial}{\partial u} - 2uv\frac{\partial}{\partial v}$ on U_S

Then V_2 only vanishes at N.

Q3) $\dim K = k$, $\dim L = l$, $\dim M = m$.

$\forall p \in K$ $\exists$ a chart $(\overset{p}{\cap}\!\mathcal{U}, \phi)$ in K, $(\overset{p}{\cap}\!V_1, \psi_1)$ in L

st. $\psi_1\big|_{V_1 \cap K} = i_1 \circ \phi$ where $i_1 : \mathbb{R}^k \to \mathbb{R}^l$, $x \mapsto (x, 0)$

and $\mathcal{U} = V_1 \cap K$.

Since $p \in K \subset L$, $\exists$ chart $(\overset{p}{\cap}\!V_2, \psi_2)$ in L, $(\overset{p}{\cap}\!W, \eta)$ in M

st. $\eta\big|_{W \cap L} = i_2 \circ \psi_2$ where $i_2 : \mathbb{R}^l \to \mathbb{R}^m$, $x \mapsto (x, 0)$

where $V_2 = W \cap L$

Let $V = V_1 \cap V_2$, WLOG we can replace V_1, V_2 by V in our setting since $\psi_1 \circ \psi_2^{-1} : \mathbb{R}^l \to \mathbb{R}^l$ is a diffeo.

we can extend it to $T : \mathbb{R}^m \to \mathbb{R}^m$ st. $T = (\psi_1 \circ \psi_2^{-1}) \times id$.

Now $(W, T \circ \eta)$ is a coord. chart on $W \subset M$. and

$$T \circ \eta\big|_{W \cap L} = T\big|_{\mathbb{R}^l \times \{0\}} \circ (i_2 \circ \psi_2) = i_2 \circ \psi_1$$

$$\Rightarrow \; T \circ \eta\big|_{W \cap K} = i_2 \circ i_1 \circ \phi.$$

by shrinking W we can assume $W \cap K \subset \mathcal{U}$, done

3-1)

$p \in M$, take $(\mathcal{U}, \phi)$ coordinate of M around p.

let $\phi = (x_1, \cdots, x_n)$

$F_* = 0 \Rightarrow \forall f \in C^\infty(N)$, $F_*\left(\frac{\partial}{\partial x_i}\right)(f) = 0 \quad \forall i$.

$\Rightarrow \frac{\partial}{\partial x_i}(f \circ F \circ \phi^{-1}) = 0$

$\Rightarrow f \circ F$ is const. on $\mathcal{U}$ for all $f \in C^\infty(N)$

$\Rightarrow F$ is constant on $\mathcal{U}$, otherwise if $\exists$ two distinct points in $F(\mathcal{U})$, then we can take $f \equiv 1$ near one point and $f \equiv 0$ near the other point, then $f \circ F$ is not const. on $\mathcal{U}$.

Now F is locally const., and M is connected, so F is constant on M //

3-2) The mapping α is clearly linear, to check that α^{-1} is given by $\alpha^{-1}(X_1, \cdots, X_k) = (j_{1*}X_1, \cdots, j_{k*}X_k)$, just note that $\forall q \in M_n$, $(1 \le n \le k)$ $\pi_m \circ j_n(q) = p_m$ if $m \neq n$ and $\pi_n \circ j_n(q) = q$.

also $j_n \circ \pi_n = id$.

3-3, Let $F: N^n \xrightarrow{\text{diffeo}} M^m$ with $N \neq \phi$

so $\exists p \in N$. Now $F_*|_p : T_p N \longrightarrow T_{F(p)} M$

is a vector space isom. and we know $\dim T_p N = n$,

$\dim T_{F(p)} M = m$, so as an isomorphic vector spaces,

they must have to same dimension, hence $m = n$.