Bonus Problem Solution

$F : \mathbb{R}^n \to S(n)$

$F(p)$ has distinct eigenvalues $\lambda_1 < \dots < \lambda_n$.

Set $G(x,t) = \det(F(x) - tI_n)$.

Apply the Implicit Function Thm to G at the point(s) (p, λ_i).

$$\frac{\partial G}{\partial t}(p,\lambda_i) = \left.\frac{d}{dt}\right|_{t=\lambda_i} \det(F(p) - tI_n)$$

$$= \left.\frac{d}{dt}\right|_{t=\lambda_i} \prod_{j=1}^{n} (\lambda_j - t)$$

$$= \left.\sum_k \prod_{\substack{j=1 \\ j\neq k}}^{n} (\lambda_j - t)(-1)\right|_{t=\lambda_i}$$

$$= (-1) \prod_{\substack{j=1 \\ j\neq i}}^{n} (\lambda_j - \lambda_i)$$

$\neq 0$ since distinct.

So by the Implicit Function Thm there are nbhds W_i of p in $\mathbb{R}^n$ and ~~function~~ smooth maps $F_i : W_i \to \mathbb{R}$ such that

$$G(x, F_i(x)) = 0 \quad \text{for all } x \in W_i$$

$$\Updownarrow$$

$$\det(F(x) - F_i(x) I_n) = 0 \quad \text{for all } x \in W_i$$

$$\Updownarrow$$

$F_i(x)$ is an eigenvalue of $F(x)$

- Set $\hat{U} = \bigcap_{i=1}^{n} W_i$
- At $p \in \hat{U}$ $F_i(p) = \lambda_i$ are distinct.
- By continuity we can choose an open $U \subset \hat{U}$ s.t.

$F_i(x) = F_j(x) \Leftrightarrow i = j \quad \forall x \in U$.