Math 481 Introduction to Differential Geometry

Test 2, Tuesday April 28, 2009

NAME:  

Key

Please read all the questions carefully. You must show all of your work to earn full credit.
1. (15 points) Let \((U, \phi)\) and \((V, \psi)\) be charts on a two-dimensional manifold \(M\) such that \(U \cap V \neq \emptyset\). Denote the coordinates on the copy of \(\mathbb{R}^2\) containing \(\phi(U)\) by \((x_1, x_2)\), and the coordinates on \(\psi(V)\) by \((y_1, y_2)\).

(a) Let \(W\) be a \((0,2)\)-tensor field on \(M\) whose coordinate representative in \(\phi(U)\) is

\[
W^U(x_1, x_2) = (x_1 + 5) dx_1 \otimes dx_1.
\]

If \(y_1 = x_1 + 3x_2\) and \(y_2 = -x_1\) compute \(W^V(y_1, y_2)\).

\[
W^V_{ij} = \sum_{\xi, \kappa} \frac{\partial x_\xi}{\partial y_i} \frac{\partial x_\kappa}{\partial y_j} W^{U}_{\xi, \kappa}
\]

\[x_1 = -y_1, \quad x_2 = \frac{1}{3} (y_1 + y_2)\]

\[W^V_{i1} = W^V_{12} = W^V_{21} = 0, \quad W^V_{22} = -y_2 + 5.\]

\[W^V(y_1, y_2) = (-y_2 + 5) dx_2 \otimes dy_2.\]

(b) Define what is means for a \((0,2)\)-tensor field \(\omega\) on \(M\) to be a 2-form on \(M\). Is the tensor field \(W\) above a 2-form?

\[\omega(V_1, V_2) = -\omega(V_2, V_1) \text{ for all } V_1, V_2\]

\[W^U(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}) = x_1 + 5 \neq 0 \text{ for } x_1 \neq -5.\]

So \(W\) is not a 2-form.
2. (25 points) Consider $\mathbb{R}^3$ with coordinates $(x, y, z)$.

(a) Let $\omega$ be the 1-form on $\mathbb{R}^3$ defined by

$$\omega(x, y, z) = (zx^2 + yx^2) \, dy + (zx^2 + yx^2) \, dz.$$ 

Compute $d\omega$.

$$d\omega = \left(22x + y2x\right) \, dx \wedge dy + \left(22x + y2x\right) \, dx \wedge dz + x^2 \, dy \wedge dz$$

$$= 2x (z + y) \, dx \wedge dy + 2x (z + y) \, dx \wedge dz$$

(b) Let $\gamma: [0, 1]^2 \to \mathbb{R}^3$ be the singular 2-cube defined by

$$\gamma(x_1, x_2) = (x_1 + x_2, x_2, 2).$$

Compute the singular 1-chain $\partial \gamma$.

$$\gamma_{10}(x_1) = \gamma(0, x_1) = (x_1, x_1, 2)$$

$$\gamma_{11}(x_1) = \gamma(1, x_1) = (1 + x_1, x_1, 2)$$

$$\gamma_{20}(x_1) = \gamma(x_1, 0) = (x_1, 0, 2)$$

$$\gamma_{21}(x_1) = \gamma(x_1, 1) = (x_1 + 1, 1, 2)$$

$$\partial \gamma = -\gamma_{10} + \gamma_{11} + \gamma_{20} - \gamma_{21}$$
(c) Compute $\int_{\gamma} dw$.

\[
\gamma dw = 2(x_1 + x_2)(2 + x_2) \left( \int (x_1 + x_2) x_1 \, dx_2 + \int (x_1 + x_2) \, \lambda \, 0 \right) = 2(x_1 + x_2)(2 + x_2) \, dx_1 \, dx_2
\]

\[
\int_{\gamma} dw = \int_{C_{1}} x^3 \, dw
\]

\[
= \int_{0}^{1} \int_{0}^{1} 2(x_1 + x_2)(2 + x_2) \, dx_1 \, dx_2
\]

\[
= \int_{0}^{1} 2(2 + x_2) \left[ \frac{x_2^2}{2} + x_2 x_1 \right] \, dx_2
\]

(d) Compute $f_{\theta}, \omega$.

By Stokes' Thm 1.0 we have.

\[
\int_{\sigma} \omega = \int_{\gamma} dw = \frac{31}{6}
\]

\[
= \int_{0}^{1} 2(2 + x_2) \left( \frac{1}{2} + x_2 \right) \, dx_2
\]

\[
= \int_{0}^{1} \left( 2 + 5x_2 + 2x_2^2 \right) \, dx_2
\]

\[
= \left[ 2x_2 + \frac{5}{2} x_2^2 + \frac{2}{3} x_2^3 \right]_0^1
\]

\[
= 2 + \frac{5}{2} + \frac{2}{3}
\]

\[
= \frac{31}{6}
\]
3. (15 points) Let $X$ and $Y$ be vector fields on a manifold $M$. Recall that we have two notations for the Lie derivative of $Y$ with respect to $X$, $\mathcal{L}_X Y$ and $[X,Y]$.

(a) If $p$ is a point in $M$, define the quantity $\mathcal{L}_X Y(p)$.

$$\mathcal{L}_X Y(p) = \lim_{t \to 0} \frac{(\phi_t)_* Y(\phi_t(p)) - Y(p)}{t} = \frac{d}{dt} \bigg|_{t=0} (\phi_t)_* (X(\phi_t(p)))$$

Here $\phi_t$ is the flow of $X$ near $p$.

(b) Compute $[X,Y]$ for the following two vector fields on $\mathbb{R}^2$:

$$X = (x_1 x_2) \frac{\partial}{\partial x_1}$$

and

$$Y = x_1 \sin(x_2) \frac{\partial}{\partial x_1} + x_2 \sin(x_1) \frac{\partial}{\partial x_2}.$$

$$X = \sum v_i \frac{\partial}{\partial x_i} \quad Y = \sum w_j \frac{\partial}{\partial x_j}$$

$$[X,Y] = \sum_j \left( \sum_i v_i \frac{\partial w_j}{\partial x_i} - w_j \frac{\partial v_i}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

$$= -x_1 x_2 (\sin x_1 + \sin x_2) \frac{\partial}{\partial x_1} + x_1 x_2 \cos x_1 \frac{\partial}{\partial x_2}.$$
4. (20 points) Consider the two dimensional sphere

\[ S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} . \]

Let \( \omega \) be the 2-form on \( S^2 \) defined by restricting to \( S^2 \) the following form on \( \mathbb{R}^3 \)

\[ x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy . \]

The form \( \omega \) is nonvanishing and hence determines an orientation on \( S^2 \).

(a) Consider the singular 2-cube \( \gamma : [0,1]^2 \to S^2 \) defined by

\[ \gamma(x_1, x_2) = \left( \frac{x_1}{2}, \frac{x_2}{2}, \sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}} \right) . \]

Determine whether or not \( \gamma \) is orientation preserving.

**Note** \( \gamma^* dx = \frac{1}{2} \, dx_1 \), \( \gamma^* dy = \frac{1}{2} \, dx_2 \)

\[ \gamma^* dz = \frac{-x_1}{\sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}}} \, dx_1 + \frac{-x_2}{\sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}}} \, dx_2 . \]

\[ S_\gamma \gamma^* \omega = -\frac{x_1^2}{16} \frac{1}{\sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}}} \, dx_2 \wedge dx_1 \]

\[ -\frac{x_2^2}{16} \frac{1}{\sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}}} \, dx_2 \wedge dx_1 \]

\[ + \frac{1}{4} \sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}} \, dx_1 \wedge dx_2 . \]

\[ = \frac{1}{4} \frac{1}{\sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}}} \, dx_1 \wedge dx_2 . \]

Since this function is positive, \( \gamma \) is orientation preserving.
(b) Compute \( \int_V \omega \) where \( V \) is the image of the singular 2-cube \( \gamma \) from part (a). You can leave your answer in the form of a definite double integral.

By (a) we have:

\[
\int_V \omega = \int_{\gamma} \omega = \int_{[0,1]^2} \gamma^* \omega
\]

\[
= \int_0^1 \int_0^1 \frac{1}{\sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}}} \, dx_1 \, dx_2.
\]
5. (15 points) Consider $\omega \in \Lambda^r(M)$ and $\eta \in \Lambda^s(M)$ for $r, s \geq 1$.

(a) Prove that if $d\omega = 0$ and $d\eta = 0$ then $d(\omega \wedge \eta) = 0$.

\[
\begin{align*}
\quad d (\omega \wedge \eta) & = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta \\
& = 0 \wedge \eta + (-1)^r \omega \wedge 0 \\
& = 0
\end{align*}
\]

(b) Prove that if $\omega = d\alpha$ and $d\eta = 0$ then $\omega \wedge \eta = d\xi$ for some $\xi \in \Lambda^{r+s-1}(M)$.

choose $\xi = \alpha \wedge \eta$.

\[
\begin{align*}
d\xi & = d(\alpha \wedge \eta) = d\alpha \wedge \eta + (-1)^{r-1} \alpha \wedge d\eta \\
& = \omega \wedge \eta + 0 \\
& = \omega \wedge \eta
\end{align*}
\]

(c) Find a 2-form $\omega$ on $\mathbb{R}^4$ such that $\omega \wedge \omega \neq 0$.

\[
\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4
\]

\[
\begin{align*}
\omega \wedge \omega & = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + 0 + 0 + dx_1 \wedge dx_4 \wedge dx_3 \wedge dx_2 \\
& = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + (-1)^4 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \\
& = 2 \ dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \\
& \neq 0.
\end{align*}
\]