1) \( f: S \rightarrow \mathbb{R} \) attains its maximum at \( q \).

\[ f(q(t)) \text{ attains its maximum at } t = 0 \]
(since \( q(0) = q \)).

\[ \frac{1}{\sqrt{t}} \left| \frac{d}{dt} f(q(t)) \right| = 0 \quad t = 0 . \]

\[ (df)_q(V) = 0 \quad \text{since} \quad y_x(1) = V. \]

2) a) \( i = 2 \) \( j = 4 \)

\[ \phi_4 \circ \phi_2^{-1}(x) = \phi_4(x, -\sqrt{1-x^2}) = -\sqrt{1-x^2} \]

\[ (\phi_4 \circ \phi_2^{-1})_{,a} (\gamma, x) = \frac{x}{\sqrt{1-x^2}} \frac{\gamma}{\gamma} \]

\[ \mathcal{S}_0 \prec (\phi_4 \circ \phi_2^{-1})_{,a} (\gamma, x) \]
\[ -\frac{dy}{1-y^2} \left( \frac{ax}{\sqrt{1-x^2}} \right) \]
\[ = -a \frac{x}{\sqrt{1-y^2} \sqrt{1-x^2}} \]
\[ = a \frac{1}{\sqrt{1-x^2}} \]

and
\[ x \frac{d}{dx} \left( a \frac{dy}{dx} \right) = \frac{ax}{\sqrt{1-x^2}} \left( a \frac{dy}{dx} \right) \]
\[ = a \frac{1}{\sqrt{1-x^2}} \checkmark \]

etc.

2b) \( \alpha(p) = 0 \iff \alpha \circ (\phi_i(p)) = 0 \) for \( p \in U_i \).

But none of the local representatives \( \alpha_i \) vanish on their domains.

So \( \alpha(p) \neq 0 \) for all \( p \in S \).
If there is a function $f : S^1 \to \mathbb{R}$ such that $x = df$ then by (2) $x$ must vanish at some point in $S^1$. But we just proved that $x$ never vanishes.

3) Consider a linear combination

$$T = \sum_{i,j} T^i_j e_i \otimes \sigma^j$$

such that $T = 0$.

We must prove that $T^i_j = 0$ for $i,j) = 1, 2$.

$$T = 0 \text{ in } T_{(1)}(E)$$

$$\iff T(\sigma^k, e_\ell) = 0 \text{ for all } k, \ell = 1, 2$$

$$\iff T^k_\ell = 0 \text{ for all } k, \ell = 1, 2.$$
4) \( A^u = u_1^2 u_2 \frac{1}{2} \frac{\partial u_1}{\partial u_2} \).

\[(A^u)_1 = u_1^2 u_2, \quad (A^u)_2 = u_1^2 u_2 \]

\[(A^u)^1_1 = (A^u)^2_1 = (A^u)^2_2 = 0 \]

\[\Phi^{-1}(u_1, u_2) = \Phi([u_1: u_2: 1]) = \left( \frac{u_1}{u_2}, \frac{1}{u_2} \right) \]

\[\Phi^{-1}(w_1, w_2) = \Phi([w_1: 1: w_2]) = \left( \frac{w_1}{w_2}, \frac{1}{w_2} \right) \]

\[(A^v)_1 = (A^u)_1 \frac{\partial w_1}{\partial u_1} \frac{\partial u_2}{\partial w_1} = u_1^2 u_2 \left( \frac{1}{u_2} \right) \cdot 0 = 0 \]

\[(A^v)_2 = (A^u)_2 \frac{\partial w_1}{\partial u_1} \frac{\partial u_2}{\partial w_2} = u_1^2 u_2 \left( \frac{1}{u_2} \right) \left( -\frac{1}{w_2^2} \right) \]

\[= -\frac{u_1^2}{w_2^2} \]

\[= \frac{w_1}{w_2^4} \]
\[(A^u)^2 = (A^u)^1_2 \frac{\partial w_2}{\partial u_1} \cdot \frac{\partial u_2}{\partial w_1} = 0\]

\[(A^u)^2 = (A^u)^1_2 \frac{\partial w_2}{\partial u_1} \frac{\partial u_2}{\partial w_2} = 0\]

\[A^v (w_1, w_2) = -\frac{w_1^2}{w_2^4} \frac{\partial}{\partial w_1} \otimes dw_2.\]